Majority Stable Production Equilibria: A Multivariate Mean Shareholders Theorem

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Abstract

In a simple parametric general equilibrium model with S states of nature and $K \leq S$ firms — and thus potentially incomplete markets — , rates of super majority rule $\rho \in [0;1]$ are computed which guarantee the existence of ρ–majority stable production equilibria: within each firm, no alternative production plan can rally a proportion bigger than ρ of the shareholders, or shares (depending on the governance), against the equilibrium. Under some assumptions of concavity on the distributions of agents' types, the smallest ρ are shown to obtain for announced production plans whose span contains the ideal securities of all K mean shareholders. These rates of super majority are always smaller than Caplin and Nalebuff (1988, 1991) bound of $1 - 1/e \approx 0.64$. Moreover, simple majority production equilibria are shown to exist for any initial distribution of types when $K = S - 1$, and for symmetric distributions of types as soon as $K \geq S/2$.

Keywords: Shareholders' vote, general equilibrium, incomplete markets, super majority.

JEL Classification Number: D21, D52, D71, G39.

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1 Introduction

In this paper, a simple parametric general equilibrium model with S states of nature and $K \leq S$ firms — and thus potentially incomplete markets — is studied. There is only one good, and the agents (consumers/shareholders) are characterized by utility functions exhibiting some quadratic feature and indexed by a probability vector π in the $(S-1)$-dimensional simplex, Δ_S, that we call the type of the agent. Agents' types are supposed to be distributed, according a continuous measure with density f over Δ_S, and are only endowed with initial shares of the K firms. Since there is no consumption in period zero, firms are taken to be assets which allocate a certain mass of the good across states in period one.

Rates of super majority rule ρ are computed which guarantee the existence of ρ-majority stable production equilibria. The interpretation follows. Given initially announced production plans, a general equilibrium is computed: agents choose their optimal portfolio given the market prices, and equilibrium prices for shares occur that clear the markets. This production equilibrium is shown to be ρ-majority stable in the natural following sense: within each firm, the production plans of other firms remaining fixed, no alternative production plan can rally a proportion bigger than ρ of the shareholders, or shares, against the equilibrium.

These rates of super majority rule are computed (1) under various governances, both of the 'one person-one vote' and 'one share-one vote' types, and (2) when the considered shares are the initial (pre-trade) shares or the equilibrium (post-trade) shares. Conditions are given under which these rates are smaller than Caplin and Nalebuff (1988, 1991) bound of 64%. Moreover, it is shown that simple majority production equilibria exist for any initial distribution of types when $K = S - 1$, and for symmetric distributions of types as soon as $K \geq S/2$. Thus, even with a high degree of market incompleteness, a production equilibrium exists against which, within each firm, no alternative production plan can rally more than half of the shareholders, or shares.

The early motivation of this paper is to study whether collective choice mechanisms among the society of shareholders — and in particular the simplest one: majority voting — can help defining or qualifying the objective of the firm in a context of incomplete markets. The latter concept has received a lot of interest in the recent years [see, e.g., Citanna and Villanacci (1997), Dierker, Dierker and Grodal (1999) and Bettzüge and Hens (2000)]. In the present setup, the objective of a firm is not investigated from the perspective of efficiency or maximization of some shareholder's value or profit function [as in Drèze

(1974), Grossman and Hart (1979)], but from the point of view of stability with respect to collective decision making among shareholders [as in Drèze (1987, 1989), DeMarzo (1993)], under different types of governance.

The results proposed tend to show that market equilibria exist which are stable with respect to simple and quite operational collective decision mechanisms (here: voting rules with reasonable rates of super majority), even when the degree of market incompleteness can be considered `high'. Moreover the less incomplete the markets the smaller the rate of super majority necessary to guarantee the existence of stable general equilibria. Although these intuitive findngs are obtained in a simple setup, it is certainly valuable to have positive results of robust existence of majority majority stable production equilibria. Especially given the fact that the Social Choice literature is perceived as being dominated by impossibility results and considered useless for a general theory of decision in firms.

In standard general equilibrium models of production in a context of incomplete markets [see, e.g., Magill and Quinzii (1996), Duffie and Shafer (1988) and Geanakoplos, Magill, Quinzii and Drèze (1990)], the financial structure is usually more complex than the one presented here. And the difficulty in defining an objective function for a firm stems from the fact that, at equilibrium, shareholders can disagree on the present value of the production plans that are not in the span of the financial structure: to discount future income streams, they use shadow prices that can be different. These shadow prices are endogeneous whereas in the present paper, they are basically always pointing toward the ideal security which is exogeneously fixed, by assumption on the utility functions.

There is nevertheless a way the present paper can shed some light on the debate on which objective function the firm should optimize in the context of incomplete markets. Firm should make choices that are supported by shareholders, and the most commonly suggested behavior for the firm is that it should use the average of the shareholders' normalized present value vector, where the weights for averaging are the shares of shareholders: a `mean' shareholder is thus defined for each firm. If the latter shares are the initial shares, it is the Grossman-Hart criterion, if they are the equilibrium shares, it is the Drèze criterion. The present paper gives some insights that these two criteria are likely to give rise to majority stable production equilibria (see Section 4). The main result of this paper is that there exist production equilibria such that the K mean shareholders[1] can exactly span their type and generate their ideal security (the one they would demand if markets were complete); moreover these are the most stable equilibria. It is worth noticing

[1] Of course, in Drèze's case, as opposed to Grossman-Hart's, the mean shareholder is endogeneously determined at equilibrium.

that the assumptions under which this result holds are weaker in the case of a governance à la Drèze.

This result has no direct link with the above-mentioned criteria since the announced production plan of a firm does not have to be the optimal production plan of its mean shareholder. But the collection of K production plans (called a multiplan) should be such that their span contains the ideal security of all mean shareholders; in some way the multiplan is optimal for the K mean shareholders. Then the production equilibria are stable for the lowest possible rates of super majority. Lastly, the present paper does not study the question of optimality or constrained optimality of the stable equilibria it describes, a subject lying at the core of the literature on production in a context of incomplete markets. Especially, it does not pursue the study of Dierker, Dierker and Grodal (1999) on the relation between majority voting and welfare considerations[2].

Technically, the main results of the present paper are based on those in Caplin and Nalebuff (1988, 1991). Indeed, the case where agents are distributed over Δ_S and there is only one firm ($K = 1$, and then no exchange of shares), is a sub-case of Caplin and Nalebuff (1988, 1991). And of course we get here: $\rho = 1 - 1/e \approx 0.632$. But although some assumptions are less general than those in Caplin and Nalebuff (1988, 1991), the setup is different, and more general in at least one dimension[3]. It is more general to the extent that the number of assets can be bigger than one. It is different to the extent that there is an upstream market mechanism, with equilibrium prices clearing markets for shares. Consequently there is an endogeneous allocation of shares and therefore an endogeneous distribution over types for governances à la Drèze. In the present setup, the collective choice mechanism is intertwined with a general equilibrium market mechanism.

The paper is organized as follows. Section 2 introduces the model and provides some preliminary results founding the analysis. Section 3 focuses on the canonical case where agents are described through characteristics that are uniformly distributed over Δ_S; exact computations are provided illustrating how the less incomplete the markets the smaller the required rates of super majority. Section 4 discusses the generalization of the results obtained in the previous section: Caplin and Nalebuff (1988, 1991) general upper bound of 64% for the rate of super majority is shown to hold in case the distributions of characteristics fulfill some conditions of concavity (Proposition 3 and Theorem 3); simple majority

[2] Dierker, Dierker and Grodal (1999) show through an example that majority voting and welfare considerations can be completely unrelated.

[3] Actually, Caplin and Nalebuff (1991) gives, as an illustration for a possible application of their theory, the example of voting among shareholders in a context of incomplete markets.

stable production equilibria are shown to exist under some assumptions of symmetry of the distributions of characteristics (Proposition 2) or when the degree of market incompleteness is just one (Theorem 2). Appendix A proposes some comments; in particular, through parametric examples, these rates are shown to decrease with the homogeneity of the shareholders' types, and to increase with the shareholders' pessimism. All technical proofs are gathered in Appendix B and Appendix C.

2 The model

Consider an economy with two periods, $t = 0, 1$ and S states of nature in period 1, indexed by s, $s = 1, \ldots, S$. There is one good, and a continuum of agents, each agent is indexed by probability vector $\pi = (\pi^s)_{s=1}^S$ which will be interpreted as his ideal security once the utility functions ar introduced. The agent's type π is thus taken in the $(S-1)$-dimensional simplex:

$$\Delta_S = \left\{ \pi = (\pi^1, \pi^2, \ldots, \pi^S) \in \mathbb{R}_+^S \;\middle|\; \sum_{s=1}^{S} \pi^s = 1 \right\}.$$

Agents' types are assumed to be distributed over Δ_S according to a continuous, atomless density function $f : \Delta_S \longrightarrow \mathbb{R}_+$. Consumption takes place in period one but must be decided in period zero. Agent π is characterized by a utility function: $U_\pi[x(\pi)]$, where $x(\pi) = [x^1(\pi), \ldots, x^S(\pi)]$ is agent πs consumption in period 1. Since there is only one good, it will be sometimes better to give a financial interpretation to $x(\pi)$ as an income vector. Utility functions are of a quadratic/`euclidean' type, described at the end of this section.

There are K firms indexed by k, $k = 1, \ldots, K$. All firms have the same production technology, represented for the simplicity of the analysis by the span of Δ_S:

$$\langle \Delta_S \rangle = \left\{ y = (y^1, y^2, \ldots, y^S) \in \mathbb{R}^S \;\middle|\; \sum_{s=1}^{S} y^s = 1 \right\}.$$

Agent π is endowed with initial shares of the K firms: $\theta^0(\pi) = [\theta_k^0(\pi)]_{k=1}^K$. He is then totally characterized by the vector $[\pi, \theta^0(\pi)]$. The function $\theta^0 : \Delta_S \longrightarrow \mathbb{R}_+^K$ is taken continuous and positive over Δ_S.

A firm is basically an asset which allocates an initial mass $\Gamma_k^0 = \int_{\Delta_S} f(\pi)\, \theta_k^0(\pi)\, d\pi$ of the good across states in period 1. We do not normalize it to one to allow different firms to be of different `sizes': the yield, in terms of consumption/income, of firm k in period 1 in case state s occurs is: $\Gamma_k^0\, y_k^s$. To avoid some minor technical difficulties, it is preferable not to impose sign constraints on production plans; this is fine within the financial interpretation of the model. Although it is abusive to talk about firms in such a simple framework, and better to talk about securities, we stick to this terminology and rely on the forgiveness of the reader.

Maximization program of the agents

Given an announced production plan y_k by each firm (hence an announced multi-plan $Y = (y_k)_{k=1}^K$, where all y_k's are taken different) and a vector of prices $q = (q_k)_{k=1}^K$ for the shares, each agent maximizes his utility by choosing the optimal vector of shares[4] $\theta(\pi) = [\theta_k(\pi)]_{k=1}^K$ and the optimal consumption plan $x(\pi)$ according to the maximization program $M(\pi)$:

$$\max_{[\theta(\pi);x(\pi)]} \quad U_\pi[x(\pi)]$$

$$\text{s. t.} \quad \sum_{k=1}^K q_k \left[\theta_k(\pi) - \theta_k^0(\pi)\right] = 0 \tag{1}$$

$$\text{and} \quad x(\pi) = \sum_{k=1}^K \theta_k(\pi)\, y_k \tag{2}$$

This is of course equivalent to $\tilde{M}(\pi)$:

$$\max_{\theta(\pi)} \quad \tilde{U}_{\pi,Y}[\theta(\pi)]$$

$$\text{s. t.} \quad \sum_{k=1}^K q_k \left[\theta_k(\pi) - \theta_k^0(\pi)\right] = 0$$

where $\tilde{U}_{\pi,Y}[\theta(\pi)] = U_\pi\left(\sum_{k=1}^K \theta_k(\pi)\, y_k\right)$.

Majority Stable Production Equilibrium

Given the individual demand functions for shares, an equilibrium price will clear the market for shares.

Definition 1 A Production Equilibrium (PE) is a vector $E = (Y; q; \theta(\pi))$ such that individual optimization (C_1), and market clearing (C_2), are satisfied:

(C_1) Given $(Y; q)$, for all π $[\theta(\pi)]$ solves the maximization program $\tilde{M}(\pi)$;

(C_2) For all k, $\int_{\Delta_S} f(\pi)\, \theta_k(\pi)\, d\pi = \int_{\Delta_S} f(\pi)\, \theta_k^0(\pi)\, d\pi \left(= -\theta_k^0\right)$.

[4] The choice has been made here not to impose short-sell constraints on the θ's. The aim is to prove existence of majority stable production equilibria, and the paper is mostly going to focus on equilibria such that $\theta(\pi) > 0$ for all π

For a firm k, given a PE E, a distribution of voting weights $\eta : \Delta_S \to \mathbb{R}^K_+$ ($\eta \equiv \theta^0$ or θ), and two production plans (y_k, z_k), denote $I_{E,\eta}(y_k)$ the subset of agents π endowed with a positive voting weight[5] in firm k (i.e., agents such that $\eta_k(\pi) \geq 0$), and denote $I_{E,\eta}(z_k, y_k)$ [$\subset I_{E,\eta}(y_k)$] the subset of agents π endowed with a positive voting weight in firm k who prefer z_k to y_k, i.e., such that

$$\eta_k(\pi) \geq 0 \quad \text{and} \quad U_\pi[x(\pi) + \theta_k(\pi)(z_k - y_k)] \geq U_\pi[x(\pi)]\,;$$

where $x(\pi)$ is defined through equations (2). Define:

$$P_{E,\eta}(z_k, y_k) = \frac{\int_{I_{E,\eta}(z_k,y_k)} f(\pi)\, d\pi}{\int_{I_{E,\eta}(y_k)} f(\pi)\, d\pi} \quad \text{and} \quad A_{E,\eta}(z_k, y_k) = \frac{\int_{I_{E,\eta}(z_k,y_k)} f(\pi)\, \eta_k(\pi)\, d\pi}{\int_{I_{E,\eta}(y_k)} f(\pi)\, \eta_k(\pi)\, d\pi}\,;$$

respectively the fraction of shareholders (with voting rights) and the fraction of vote shares who prefer z_k to y_k. Define moreover

$$P_{E,\eta}(y_k) = \sup_{z_k \in \Delta_S} P_{E,\eta}(z_k, y_k) \quad \text{and} \quad A_{E,\eta}(y_k) = \sup_{z_k \in \Delta_S} A_{E,\eta}(z_k, y_k)$$

the maximal fractions (resp. of the shareholders/shares, with voting rights) against y_k.

Definition 2 For any real $\rho \in [0, 1]$, a ρ–Majority Stable Production Equilibrium under

- the 'one person-one vote, pre-trade' governance (in short, a ρ–MSPEp0) is a PE E such that for all k, $P_{E,\theta^0}(y_k) \leq \rho$;
- the 'one person-one vote, post-trade' governance (ρ–MSPEp1) is a PE E such that for all k, $P_{E,\theta}(y_k) \leq \rho$;
- the 'one share-one vote, pre-trade' governance (ρ–MSPEa0), is a PE E such that for all k, $A_{E,\theta^0}(y_k) \leq \rho$;
- the 'one share-one vote, post-trade' governance (ρ–MSPEa1), is a PE E such that for all k, $A_{E,\theta}(y_k) \leq \rho$.

For $\rho = 1/2$, such an equilibrium is a simple–Majority Stable Production Equilibrium (or s-MSPE).

[5] Only such agents have the right to vote in the present setup.

Remark: The p0 and p1-governance are not distinct as soon as everybody is positively endowed with shares of all firms, both initially and at equilibrium. This will be mostly the case in the present paper. It is clear that the most interesting governance is the a1-governance. Nevertheless, there is some difficulty in defining a ρ^*-Majority Stable Production Equilibrium for the a1-governance since the number of post-trade shares with voting rights, $\int_{I_{Y;\theta}(y_k)} f(\pi)\, \theta_k(\pi)\, d\pi$ is endogeneous and can be bigger than the initial allocation of shares, Γ^0_k, in case part of the agents choose to be short on k's stock market[6]. But we will concentrate in this paper on production equilibria where all agents are allocated positive post-trade shares. For other production equilibria, one can consider that the excess number of shares is allocated in a continuous way (i.e., according to f and θ^0) to all other shareholders, which does not introduce much distorsion in the model.

The concept of ρ^*-majority stable equilibrium (for $K = 1$) is linked to the Simpson-Kramer min-max majority [see Simpson (1969), Kramer (1977)]. In the present paper the concept is built to hold for $K \geq 1$: min-max majorities for production equilibria are (resp., for each governance):

$$\rho^*_{p0} = \inf_{PE\,(Y;q;\theta)} \max_k P_{Y;\theta^0}(y_k) \quad , \quad \rho^*_{p1} = \inf_{PE\,(Y;q;\theta)} \max_k P_{Y;\theta}(y_k)$$

$$\rho^*_{a0} = \inf_{PE\,(Y;q;\theta)} \max_k P_{Y;\theta^0}(y_k) \quad \text{and} \quad \rho^*_{a1} = \inf_{PE\,(Y;q;\theta)} \max_k P_{Y;\theta}(y_k) \quad .$$

Assumptions on the utility functions U_π

The utility functions U_π are defined on R^S and assumed to satisfy the two following sets of assumptions:

- Assumption (A) : U_π is increasing, strictly quasi concave, continuously differentiable and homothetic;
- Assumption (E) : The indifferences surfaces of U_π cut $\langle \Delta_S \rangle$ through hyperspheres centered on π

Taking homothetic utility functions will allow to focus on consumptions in $\langle \Delta_S \rangle$ (since we'll only consider PE with[7] $q = 1_K$, see next subsection). Assumption (E) (said to be the `euclidean' assumption) is more problematic: it is standard in Social Choice theory,

[6] In fact, the stock repurchase plans that some firms implement might be considered as introducing some type of endogeneity in the total numbers of shares.

[7] Notation: $q = 1_K$ stands for $q_k = 1$, all k.

and taken for purely technical reasons. The motivation behind this assumption is the following: when asked whether they agree with an infinitesimal change[8] $u \in R^S$ in the production plan of firm k, indifferent shareholders should be on a hyperplane in Δ_S. It is nevertheless clear that such utility functions exhibit some form of quadratic feature, an such features are regularly assumed in the finance literature, e.g., in the CAPM.

When there is only one firm ($K = 1$) as in Caplin and Nalebuff (1988, 1991), it is enough to take utility functions of the separable form:

$$U_\pi[x(\pi)] = \sum_{s=1}^{S} \pi^s v^s[x^s(\pi)]\,: \tag{3}$$

In that case, the type π is the subjective probability of the agent over states of nature. The fact that the elementary utility functions are common across the population secures the needed condition [see Grandmont (1978)]. The reason is simple to see: when $K = 1$, $x(\pi) = y_1$ is independent of π and for any infinitesimal change $u \in R^S$ in the production plan, shareholders indifferent to the proposed change are described by the equation $\sum_s \pi^s u^s D v^s[y_1] = 0$ which defines a hyperplane. If $K > 1$, shareholders indifferent to an infinitesimal change u in the production plan of firm k are described by the equation $\sum_s \pi^s u^s D v^s[\bar{x}^s(\pi)] = 0$, where $\bar{x}^s(\pi)$ stands for the optimal consumption of agent π and differs across π For instance, in the log-linear case where $v^s \equiv \ln$, the latter equation almost never defines a hyperplane in Δ_S. But some of the results proposed in the paper are valid with utility functions of the form (3); this discussion is posponed to Section 4.4.

A last difficulty is to avoid negative consumptions/incomes. We basically discard this problem: (i) in case the utility functions are of the separable form (3), by assuming that v^s satisfies the Inada conditions: $\lim_{x_i \to 0} D v^s(x) = +\infty$; (ii) in case the utility functions satisfy assumption (E), by endowing the agents with an appropriate quantity, $x^0(\pi)$, of the consumption good, whatever the occuring state of nature[9].

The Pareto criterion

Among all production equilibria, we will restrict our attention to those that respect the Pareto criterion: an eligible production plan for majority stability should be such that

[8] As already written in the introduction, the assumption of concavity of the individual utility functions entails that the most challenging production plans are infinitesimally close to the staus quo; see Lemma 2 in Appendix B. Therefore, a challenger is basically an infinitesimal change u in the production plan, with, given the technological constraints, $\sum_s u^s = 0$.

[9] Since we will only consider multiplans Y which spans a hyperplane having a non-empty intersection with Δ_S, a uniform upper bound can be found on $x^0(\pi)$, for all π

there does not exist an alternative production plan preferred by all shareholder endowed with a voting right (i.e., endowed with a positive quantity of shares). The following observation shows that, in the present framework, a necessary and sufficient condition is that stock prices be all equal[10].

Observation 1 A PE $(Y; q; \theta)$ satisfies the Pareto criterion if and only if $q = 1_K$.

Proof: Consider a PE $(Y; q; \theta)$ such that $q \neq 1_K$. Consider two firms, k and j, such that $q_k > q_j$; then there exists an alternative anounced production plan z_k unanimously prefered to y_k by agents positively endowed with shares of firm k. Suppose, without loss of generality, that $q_1 > q_2$. At the PE $(Y; q; \theta)$, the gradient of $U_\pi[x(\pi)]$ with respect to $\theta(\pi)$ is colinear to q. Given $q_1 > q_2$, this entails that for all π: $DU_\pi[x(\pi)] \cdot (y_1 - y_2) > 0$. Consider $z_1 = y_1 + \epsilon(y_1 - y_2)$, we then have, for ϵ small enough and for all π $U_\pi[x(\pi) + \theta_1(\pi)(z_1 - y_1)] > U_\pi[x(\pi)]$ if $\theta_1(\pi) > 0$. Hence for the 'if' part of the assertion. The 'only if' part is obviously true. $\square$

In the sequel of the paper, we'll define a Pareto production equilibrium as a PE with unit prices: $(Y; 1_K; \theta)$.

Denote $\langle Y \rangle$ the vectorial subspace, in $\langle \Delta_S \rangle$, spanned by Y. At a PE with unit prices, the optimal choice of an agent is — up to multiplication by a scalar, given assumption (A) — the point of tangency between $\langle Y \rangle$ and the sections by $\langle \Delta_S \rangle$ of the agent's indifference curves. This optimal point is the orthogonal projection of π on $\langle Y \rangle$ when assumption (E) is fulfilled.

This last property entails the following geometric interpretation, à la Caplin and Nalebuff, of the main argument of the paper (proven in Lemma 2 in Appendix B): trying to find a best challenger to y_k, within the production plans of firm k (the production plans of other firms remaining fixed), reduces to try and cut the support, Δ_S, of the agents' types by an hyperplane containing $\langle Y \rangle$ in such a way as to maximize the difference in volume of the two resulting pieces — a volume computed using the distribution of voting weights, as the governance specifies it.

[10] DeMarzo (1993) proves that a production plan which is stable with respect to a 'unanimity responsive' collective decision rule should be chosen by using a normalized present value vector in the convex hull of those of all shareholders. A 'unanimity responsive' collective decision rule is such that it should be able to implement an alternative production plan that Pareto dominates the incumbent. See also Proposition 31.3 in Magill and Quinzii (1996).

A fundamental preliminary result

It states that any vectorial subspace in $\langle \Theta_S \rangle$ can be spanned by a multiplan Y that can be associated with a PE with equal unit prices.

Lemma 1 *Under assumption (A), any multiplan $Y = (y_k)_{k=1}^K$ generates a vectorial subspace that can be supported by a production multiplan associated to a PE with unit prices: there exists a production multiplan $\tilde{Y} = (\tilde{y}_k)_{k=1}^K$, with $\tilde{y}_1 = y_1$, such that $\langle Y \rangle \equiv \langle \tilde{Y} \rangle$, and $(\tilde{Y}; 1_K; \tilde{\mu})$ is a PE. Moreover, y_1 can be chosen such that $\tilde{\mu}(\pi) > 0$ for all π*

Proof: See Appendix A. □

This fundamental Lemma allows to focus only on the span $\langle Y \rangle$ of a multiplan Y, and not on the multiplan itself. Moreover, the fact that $\tilde{\mu}(\pi)$ can be taken strictly positive for all π secures that all shareholders have the right to vote and that the considered distributions of voting weights are positive over the whole support Θ_S.

3 The canonical case

We consider the canonical case of uniform distributions of initial characteristics in the set of types Θ_S. Assumptions:

- for the p0-governance: the distribution f is uniform and $\theta_k^0(\pi) > 0$ for all k, all π;
- for the p1-governance: the distribution f is uniform and $\sum_k \theta_k^0(\pi) > 0$ for all π;
- for the a0-governance: the distribution $f \cdot \theta_k^0$ is uniform for all k;
- for the a1-governance: the distribution $f \cdot \sum_k \theta_k^0$ is uniform.

It is worth noticing that the results of the present section remain valid under the assumption of separable utility functions of the type (3) (see Claim 2 in Section 4.4) and that the preceding set of assumptions are weaker for governances à la Drèze, i.e., based on post-trade shares, than for governances à la Grossman-Hart.

3.1 Existence of MSPE

For any fixed positive integers S and K, $K \leq S$, define[11]

$$\rho_{S,K} = 1 - \left(\frac{\lfloor \frac{S-1}{K} \rfloor}{\lfloor \frac{S-1}{K} \rfloor + 1} \right)^{\lfloor \frac{S-1}{K} \rfloor} . \qquad (4)$$

Theorem 1 *Fix K and S. There always exist, in the canonical case, $\rho_{S,K}$–MSPE[12] for all governances of Definition 2. Hence $\rho^* \leq \rho_{S,K}$ for all four governances.*

When $K = 1$, there are no transaction between agents and everybody keeps its initial share of the firm; since the shares are uniformly distributed accross agents, then the four governances coincide, and the above result is a particular case of Caplin and Nalebuff (1988), which gives as a uniform upper bound: $1 - 1/e \approx 0.632$. This upper bound is approached for the present concept of majority voting equilibrium in the case where the number of assets (or firms) is negligible with respect to the number of states of the world.

In other cases the rate of super-majority rule that guarantees the existence of a MSPE is lower than this previous bound. For example, whatever the number of states of nature, if $S/3 \leq K < S/2$ [resp. $S/4 \leq K < S/3$] then a rate of 56% [resp. 60%] suffices. Another example is the following immediate corollary.

Corollary 1 *S–MSPE exist as soon as $K \geq S/2$ for all four governances.*

Thus, even with a high degree of market incompleteness, a production equilibrium exists against which, within each firm, no alternative production plan can rally more than half of the shareholders, or shares. The sequel of this section is a proof of Theorem 1 which goes through the design of the 'right' securities.

3.2 Basic construction of a MSPE

The aim is to construct a ρ–MSPE for the lowest possible ρ. For fixed S and K, define $n = \left\lfloor \frac{S-1}{K} \right\rfloor$, so that $S = nK + m$, with $1 < m \leq K$. We then construct the following partition of the set of states of nature into K subsets (according to the natural order, the m first subsets contain $n+1$ elements, the $K - m$ others contain only n elements):

$$S_k = \{(k-1)(n+1)+1, \ldots, k(n+1)\} \quad \text{for} \quad 1 \leq k \leq m$$
$$T_k = \{m + (k-1)n + 1, \ldots, m + kn\} \quad \text{for} \quad m+1 \leq k \leq K$$

[11] For any real x, we denote by $\lfloor x \rfloor$ the largest integer smaller or equal to x, and by $\lceil x \rceil$ the smallest integer larger or equal to x.

[12] In fact there is a continuum of such MSPE (see the proof).

Define the K production plans $\bar{Y} = (\bar{y}_k)_{k=1}^{K}$, such that:

$$\text{for } k \leq m;\ \bar{y}_k^s = \begin{cases} \frac{1}{n+1} & \text{if } s \in S_k \\ 0 & \text{otherwise} \end{cases};\ \text{for } k \geq m+1;\ \bar{y}_k^s = \begin{cases} \frac{1}{n} & \text{if } s \in T_k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The main argument revolves around the following proposition which is a more developed restatement of Theorem 1.

Proposition 1 *Fix S and K. Thanks to Lemma 1, there exist PE $(\tilde{Y}; 1_K; \tilde{\theta})$ that are $\pi_{S,K}$–MSPE for the four governances. They are such that $\langle \tilde{Y} \rangle \equiv \langle \bar{Y} \rangle$ and for all π, the optimal consumption is*

$$\tilde{x}(\pi) = \sum_k \tilde{\theta}_k(\pi)\, \tilde{y}_k = \sum_k \bar{\theta}_k(\pi)\, \bar{y}_k$$

where $\bar{\theta}$ is defined by[13]*:*

$$\bar{\theta}_k(\pi) = \begin{cases} \pi^{S_k} & \text{if } k \leq m \\ \pi^{T_k} & \text{if } k \geq m+1 \end{cases}. \tag{6}$$

Proof: See Appendix B. $\Box$

Example: When both f and θ^0 are taken uniform (and normalized), and all firms have the same size (Γ_k^0 is independent of K), an example of such a PE $(\tilde{Y}; 1_K; \tilde{\theta})$ is:

$$\text{for } k \leq m;\ \tilde{y}_k = \frac{(n+1)K}{S}\bar{y}_k - \frac{1}{S}\sum_{j=m+1}^{K}\bar{y}_j\ ;\ \text{and for } k \geq m+1;\ \tilde{y}_k = \bar{y}_k\ ; \tag{7}$$

$$\text{and} \quad \tilde{\theta}_k(\pi) = \begin{cases} \frac{S}{(n+1)K}\pi^{S_k} & \text{if } k \leq m \\ \pi^{T_k} + \frac{1}{(n+1)K}\pi^{S} & \text{if } k \geq m+1 \end{cases}.$$

A geometric interpretation of Proposition 1 will be helpful to understand the proof and the basic intuition of the construction. As written before, given that market equilibrium prices are 1_K, the optimal choice of an agent is the point of tangency between $\langle Y \rangle$ and the sections by $\langle \Delta_S \rangle$ of the agent's indifference curves. Denote $\tilde{\Sigma}(\pi)$ the section by $\langle \Delta_S \rangle$ of the indifference curve going through the optimal choice $\tilde{x}(\pi)$ (cf. Figure 1.a below). A change in the production plan $\tilde{y}_k$ (or equivalently $\bar{y}_k$) of firm k will then move $\langle \bar{Y} \rangle$ in

[13] Denote, for a subset V of the set of states of nature, $\pi^V = \sum_{s \in V} \pi^s$.

such a way that it still goes through all other y_j's. This change, fixing the shares at their post-trade values, projects the equilibrium consumption $x(\pi)$ inward or outward $\Sigma(\pi)$, hence resulting in an improving or impairing change of the utility level of agent π (cf. Figure 1.b below).

Lemma 2 in Appendix B shows that finding a best challenger to y_k, within the production plans of firm k (the production plans of other firms remaining fixed), amounts to finding the infinitesimal move of y_k which improves the welfare of the biggest proportion of shareholders or shares. Given assumption (E), this reduces to try and cut Δ_S by a hyperplane (orthogonal to this infinitesimal change) containing $\langle Y \rangle$ in such a way as to maximize the difference in volume of the two resulting pieces. The best infinitesimal change (of y_k) is pointing toward the largest piece. As in Caplin and Nalebuff (1988) it is shown that, when the distribution of initial characteristics is uniform, the most challenging infinitesimal change of the production plan y_{k_3} is to sacrifice one state of nature to the benefit of all others[14], and implement a change $\left(-\epsilon; \frac{\epsilon}{S-1}; \dots; \frac{\epsilon}{S-1}\right)$.

3.3 Geometric illustration: S = 3 and K = 2

In the case S = 3 and K = 2, and under the assumptions given in the example following Proposition 1, with $\bar{y}_1 = (1/2; 1/2; 0)$ and $\bar{y}_2 = (0; 0; 1)$, and therefore $y_2 = \bar{y}_2$ and $y_1 = (2/3; 2/3; -1/3)$, one gets:

$$[\tilde{\mu}_1(\pi); \tilde{\mu}_2(\pi)] = \left(\frac{3}{4}[\pi_1 + \pi_2]; \pi_3 + \frac{1}{4}[\pi_1 + \pi_2]\right) \quad \text{and} \quad [\hat{\mu}_1(\pi); \hat{\mu}_2(\pi)] = \left([\pi_1 + \pi_2]; \pi_3\right):$$

This is drawn on Figure 1.a; the indifference curve $\Sigma(\pi)$ corresponding to the optimal utility level for agent π is drawn: it is a circle around the ideal security π. An illustration of the previous discussion is now provided in this simple case and basically holds for the four governances.

Optimal cutting of the simplex: It should be clear on the drawing why $(Y; 1_2; \tilde{\mu})$ is majority stable for the simple-majority rule under all four governances. Indeed, consider, instead of y_2, another proposal y_2^0 (see Figure 1.b). The shares being fixed, the new consumption of agent π will become $x(\pi)^0$ which dives inward Σ, hence resulting in a higher utility. But for the symmetric (with respect to $\langle Y \rangle$) agent, characterized by type $\pi' = (\pi_2; \pi_1; \pi_3)$, who at equilibrium consumes the same $x(\pi') = x(\pi)$, this is an impairing change. Hence at least half of the agents (the left part of the triangle) finds it impairing that any rightward change of the production plan of y_2 be implemented. Symmetrically, any, even infinitesimal, leftward change of y_2 is going to be blocked by the agents on the right-hand side of $\langle Y \rangle$. Finally, since both agents π and π' have the same share of firm 2, it is obviously the case that the simple-majority stability property holds for the four types of governance. The same type of argument holds to prove

[14] This is actually very classical in Social Choice theory and illustrated by the problem of having to divide a pie among S individuals; whatever the initial allocation, there is a majority of $\frac{S-1}{S}$ to expropriate one individual of his share and distribute it evenly to the others.

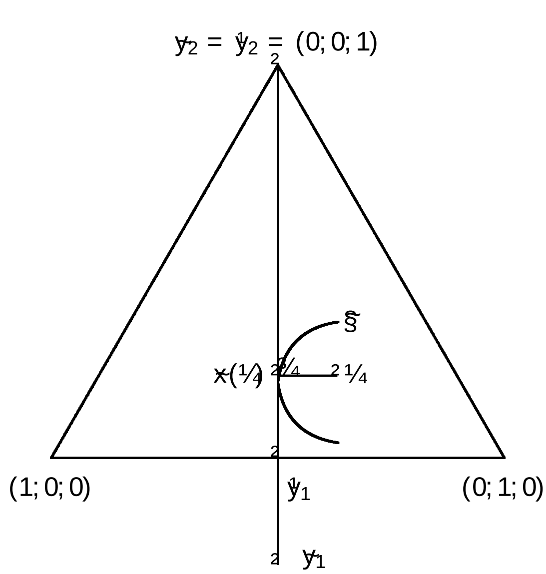

Figure 1.a

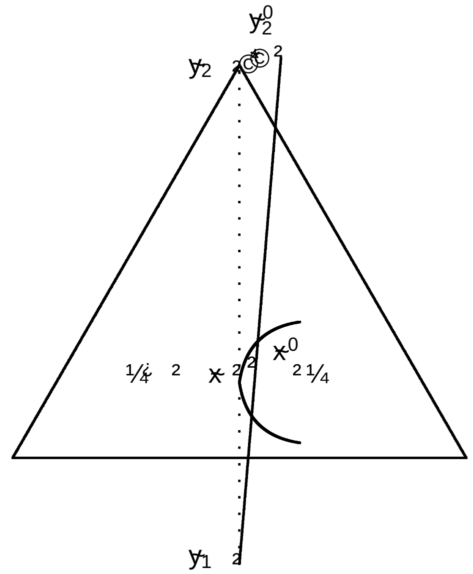

Figure 1.b

that any change in the production plan $\bar{y}_1$ is going to be blocked by at least half of the shareholders, in number and volume of shares.

Moreover it is clear that there are many ways to cut ¢$_3$ into two pieces of equal sizes. The two pieces do not have to be symmetric. Actually, Lemma 1 shows that any cutting of ¢$_3$ can be spanned by two production plans $(y_1; y_2)$ which will generate a PE with unit prices, hence securing that the fundamental geometrical interpretation of Theorem 1 be valid. This ensures a continuum of s-MSPE is the present simple case.

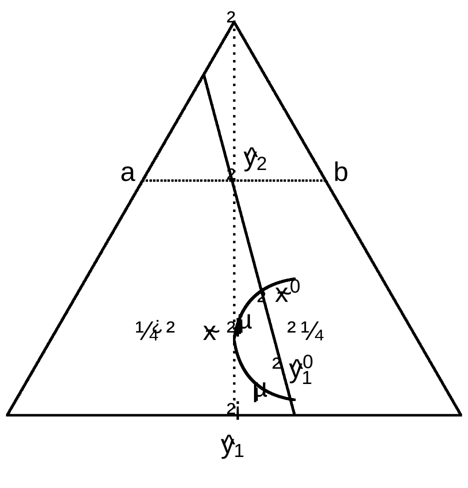

Figure 1.c

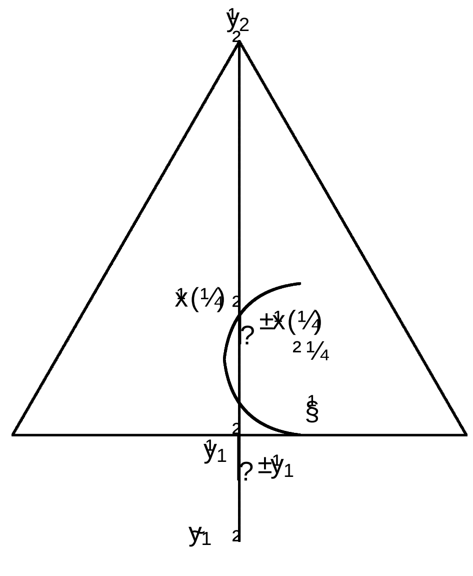

Figure 1.d

Multiplicity of the MSPE: There are a contimuun of PE that end up with the same \cutting" of ¢$_S$

with unit prices q = 1_2 for both assets, and unchanged $\hat{\mu}$ (see Lemma 1): for all ®, $y_1 = (\frac{1}{3} + ®; \frac{1}{3} + ®; \frac{1}{3} ¡ 2®)$ and $y_2 = (\frac{1}{3} ¡ ®; \frac{1}{3} ¡ ®; \frac{1}{3} + 2®)$ will always found a PE with q = 1_2 and equilibrium shares $\hat{\mu}$. (Notice ® = $\frac{1}{3}$.) For example, with ® = $\frac{1}{6}$, one gets $\hat{y}_1 = \bar{y}_1 = (1=2; 1=2; 0)$ and $\hat{y}_2 = (1=6; 1=6; 2=3)$, with: $[\hat{\mu}_1(¼); \hat{\mu}_2(¼)] = [¼ + ¼] ¡ \frac{1}{2}¼; \frac{3}{2}¼$ so that $\hat{x}(¼) = x(¼) = \bar{x}(¼)$ (see Figure 1.c). The drawn change from $\hat{y}_1$ to $\hat{y}_1^0$ will be utility improving for agent ¼ but utility impairing for agent ¼. Notice here that all agents characterized by a type ¼ such that ¼ ¸ 2=3 (i.e., above the dotted line [a; b]), do not have the right to vote under governances based on post-trade shares since their post-trade shares in $\hat{y}_1$ are negative. Hence the same rule as before is ful¯lled: any rightward (resp. leftward) change in the production plan $\hat{y}_1$ will be blocked by (at least) the left-hand (resp. right-hand) side of the triangle, whose top has been cut-o®. It is to avoid the minor and irrelevant technical di±culty of having to compute relative volumes in a cut-o® simplex that PE are contructed for which all shares allocated at equilibrium are positive (i.e., with ® > 1=3).

Assets with di®erent prices, the Pareto criterion: One can easily see that any proposed change of y_1 along the line hYi will be unanimously rejected. This fact is linked to the reason why there is no majority stable production equilibrium with announced production plans $(\bar{y}_1; \bar{y}_2)$: in fact the PE based on this multiplan does not satisfy the Pareto criterion (see Observation 1). Indeed, the equilibrium price vector is then such that $q_1 > q_2$: the shareholders will ¯nd it optimal to `load' more than in the above case their portfolio with shares of $\bar{y}_2$ (see Figure 1.d) to reach the optimal consumption $\bar{x}(¼)$. As drawn on Figure 1.d, the optimal utility level will then generate an indi®erence surface § not tangent to hYi. Given the quasi-concavity of the utility functions, any change $±\bar{y}_1$ of $\bar{y}_1$ toward y_1 will be unanimously supported, since the consecutive change $±\bar{x}(¼)$ is always utility improving. This is true untill $\bar{y}_1$ reaches y_1.

4 More general cases

In this section, more general density functions, f, and initial distributions of shares, μ^0, are investigated. To avoid minor technical di±culties that would make the reading less confortable without making the problem richer, we consider only strictly positive initial distributions of characteristics: $\mu^0(¼) > 0$ and $f(¼) > 0$ for all ¼. The aim is to generalize as much as possible the results of the previous section. In a ¯rst subsection, we investigate, for unspeci¯ed f and μ^0, the case of complete markets, along with the case of incomplete markets with only one dimension of incompleteness. Then we consider the case of symmetric distributions of characteristics (subsection 4.2). For these two cases, simple majority production equilibria are shown to exist. Finally, the case of º-concave distributions of characteristics is considered (subsection 4.3), an assumption regarded as imposing some measure of consensus in the society of shareholders. Caplin and Nalebu® (1991) results are then used to provide ratios of ½ majority stable production equilibria.

4.1 The cases $K = S$, $K = S - 1$

The case $K = S$ is trivial, since for a PE $(Y; 1_K; \theta)$ — whose existence is secured by Lemma 1— every agent of type π is able to generate its idiosyncratic ideal security: $[\sum_k \theta_k^0(\pi)] \cdot \pi$. In this case, in equilibrium, all y_k's are unanimously supported against any alternative production plan for any f and any initial distribution of shares θ^0; i.e., $P_{Y;\theta^0}(y_k) = A_{Y;\theta^0}(y_k) = P_{Y;\theta}(y_k) = A_{Y;\theta}(y_k) = 0$ for all k, as the theory of complete markets predicts. We thus have the following observation.

Observation 2 *If $K = S$, for any density f and any initial distributions of shares θ^0, there exist PE which are stable for any voting rule (even infra-majority voting rule[15]).*

The case $K = S - 1$ is more difficult and interesting. As far as the p0 and p1-governances are concerned, the argument is straighforward since the same distribution of voting weights, f, is taken for all firms. Therefore a median-voter-like argument allows to go through: For a PE $(Y; 1_K; \theta)$, we know that finding a best challenger to the announced production plan y_k amounts to cut the support of agents' types by a hyperplane containing $\langle Y \rangle$. But there is a unique such hyperplane, i.e., $\langle Y \rangle$ itself. Therefore, to prove existence of a s–MSPE, it is enough to choose $\langle Y \rangle$ such that it separates Δ_S into two pieces of equal measure with respect to f. This is obviously always possible, and there is an infinite number of ways to do so as soon as $S > 2$. Thanks to Lemma 1, we know that such a hyperplane can be supported by a PE with unit prices and positive shares.

The argument is more complicated for governances based on shares, e.g., the a0 and a1-governances. Indeed, $\langle Y \rangle$ should be chosen such that it separates Δ_S into two pieces of equal measure simultaneously with respect to K $(= S - 1)$ distributions of voting weights. Hence a 'multivariate-median-voter' argument is necessary. The following proposition, based on degree theory and using the Borsuk-Ulam theorem, is shown. [16]

Theorem 2 *If $K = S - 1$, there exist s-MSPE for any f and any θ^0, for all four governances.*

Proof: To prove existence of s–MSPEa0 one has to choose $\langle Y \rangle$ that separates Δ_S into two pieces of equal measure with respect to the distributions $f \cdot \theta_k^0$, for all k.

---

[15] An infra-majority voting rule is a majority rule with rate $\rho < 1/2$, i.e. such that an alternative a defeats an alternative b if a proportion bigger than ρ of the population prefers a to b; hence it is possible that two alternatives defeat each other at the same time.

[16] It is worth noticing that it remains valid under assumption (A) only (cf. Claim 1 in Section 4.4) on the utility functions for the governance based on pre-trade shares.

Consider the $(S-2)$–unit sphere (of dimension $S-2$) S_{S-2}. For any point φ on the sphere, denote $\langle\varphi\rangle$ the hyperplane (of dimension $S-2$) in $\langle\Delta_S\rangle$ that is orthogonal to the vector $\overrightarrow{0\varphi}$ and divides Δ_S into two pieces of equal measure with respect to the distribution $f\cdot\sum_k\theta'_k$. Denote $\langle\varphi\rangle^+$ the one of these two pieces toward which $\overrightarrow{0\varphi}$ points. For any k, $1\le k\le K-1\ (=S-2)$, denote $\mu'_k(\varphi)$ the (continuous) measure of $\langle\varphi\rangle^+$ with respect to the distribution $f\cdot\theta'_k$. A generalization of the Borsuk-Ulam theorem[17] states that there exists a point φ_0 such that for all k, $1\le k\le K-1$, one has:

$$\mu'_k(\varphi_0)=\mu'_k(-\varphi_0):$$

Therefore, given that $\langle\varphi_0\rangle=\langle-\varphi_0\rangle$, $\langle\varphi_0\rangle$ divides Δ_S into two pieces of equal measure with respect to the distributions $f\cdot\theta'_k$, for all k, $1\le k\le K-1$. Since by construction it also divides Δ_S into two pieces of equal measure with respect to $f\cdot\sum_k\theta'_k$, it does so with respect to $f\cdot\theta'_K$. Hence the proof for the a0-governance.

To prove existence of s–MSPEa1 one has to choose a hyperplane that separates Δ_S into two pieces of equal measure with respect to the distributions $f\cdot\theta_k$, for all k. The argument is more complicated because the latter distributions are endogeneously defined. Nevertheless, the argument also relies on the Borsuk-Ulam theorem applied to functions defined through another principle. This is postponed to Appendix B. $\Box$

4.2 Symmetric densities

It is possible to define more general assumptions under which simple majority stable production equilibria exist for all four governances — i.e., Corollary 1 holds true. We define symmetric distributions of types: for all permutations σ of $\{1;\dots;S\}$, if π^σ denotes the vector of probabilities: $(\pi_{\sigma(1)};\dots;\pi_{\sigma(S)})$, then for all π $f(\pi^\sigma)=f(\pi)$.

Proposition 2 Assume that f [resp. $f\cdot\theta'_k$ for all k, $f\cdot\sum_k\theta'_k$] is symmetric over Δ_S, then s–MSPEp0 and p1 [resp. s–MSPEa0, s–MSPEa1] exist as soon as $K\ge S/2$.

Proof: Thanks to Lemma 2, this goes by proving that any hyperplane through $\bar{Y}^{1}$ (as defined by equations (5)) cuts Δ_S into two equal parts, in terms of shareholders (first

[17] See Theorem 3.2.7 in Lloyd (1978): Let D be a bounded, open, symmetric subset of R^n containing 0; let $\mu:\partial D\to R^m$ be continuous, and $m<n$; then there is $\varphi\in\partial D$ such that $\mu(\varphi)=\mu(-\varphi)$. Here D is the unit ball, $n=S-1$, $\partial D\equiv S_{S-2}$, and $m=S-2$: $\mu(\varphi)=(\mu'_1(\varphi);\dots;\mu'_{K-1}(\varphi))$. An illustration is that there exist two antipodal points on the earth with same temperature and pressure. See also Guillemin and Pollack (1974), pages 91-93.

assertion of the proposition) as well as in terms of shares (second assertion). Since $K \geq S/2$, one has $S = K + m$ with $m \leq K$. To any π associate its symmetric through $\langle 1 \rangle$:

$$\pi^* = (\pi^2; \pi^1; \ldots; \pi^{2m}; \pi^{2m-1}; \pi^{2m+1}; \ldots; \pi^S) .$$

Generically, π and π^* are strictly on each side of $\langle 1 \rangle$, and then will always counter-balance each other in any collective decision making under 'one person-one vote' governances. Under the assumptions of the proposition they have the same amount of shares of each firm, and will always counter-balance each other in any collective decision making under 'one share-one vote' governances. 2

In fact, as easily seen from the proof, much lighter assumptions of symmetry can ensure the result. Indeed, the argument developed here shows some similarity with the underlying analysis in Grandmont (1978): in that paper, existence of majority-stable equilibria (in the case without exchange: $K = 1$) was shown for centrally-symmetric supports of agents' types. The present argument relies on the same principle: the simplex Δ_S is symmetric, not with respect to a point, but with respect to K-dimensional subspaces (with $K \geq S/2$), and the only needed assumption is that the distributions of characteristics be symmetric with respect to one of these subspaces[18].

4.3 ν-concave densities

A density function f is ν-concave over Δ_S if for all $\pi, \pi' \in \Delta_S$, $\forall \lambda \in [0;1]$,

$$f[(1-\lambda)\pi + \lambda\pi'] \geq [(1-\lambda)f(\pi)^\nu + \lambda f(\pi')^\nu]^{1/\nu} .$$

This assumption is regarded as imposing some measure of consensus in the society. Notice that for $\nu = \infty$, one gets the uniform distribution of Section 3. Define[19]:

$$\rho(S;\nu) = 1 - \left(\frac{S - 1 + 1/\nu}{S + 1/\nu}\right)^{S+1/\nu} .$$

Consider a PE $(Y; 1_K; \theta)$. As in the canonical case, findng a best challenger to the equilibrium production plan of a firm reduces to try and cut the support Δ_S by an

[18] There is the implicit feature, in Caplin and Nalebuff (1988), that the simplex is, as a support of voters' type, the geometrical shape that allows the most uneven cutting through the center of gravity (see the principle of symmetrization of Schwartz on which they found this feature): if an upper bound works for the simplex, it works for any other convex support. This feature might not be true anymore as far as cutting the support through a well-chosen K-dimensional subspace is concerned.

[19] The ratio $\rho(S;\nu)$ is bounded above by $1 - 1/e$ when $\nu \geq 0$.

hyperplane containing $\langle Y\rangle$ in such a way as to maximize the difference in volume of the two resulting pieces. When the distribution of shareholders' voting weights is exogeneously fixed (as for the p0, p1 and a0 governances), given that the support of all considered distributions is convex, one can directly import Caplin and Nalebuff (1991) main result on ν-concave distribution of characteristics to get the following proposition.

Proposition 3 If f is ν-concave, then for $\nu \geq -1/S$, any PE $(Y; 1_K; \theta)$ such that $\langle Y\rangle$ contains the mean shareholder's type[20] π_g of distribution f is a $\rho^*(S;\nu)$–MSPEp0 and p1.

If $f \cdot \theta_0^k$ is ν-concave for all k, then for $\nu \geq -1/S$, any PE $(Y; 1_K; \theta)$ such that $\langle Y\rangle$ contains the K mean shareholder types $(\pi_g^k)_{k=1}^K$ of the K distributions $(f \cdot \theta_0^k)_{k=1}^K$ is a $\rho^*(S;\nu)$–MSPEa0.

In both cases, there exist a continuum of such $\rho^*(S;\nu)$–MSPE.

It is clear that, for the for the 'one person-one vote' governances, the higher K, the smaller the rate of super-majority ρ that is necessary to guarantee the existence of ρ-majority stable production equilibria. Indeed, on top of having to cut Δ_S through its center of gravity, one can add as many constraints as there are firms, each added constraint lowering the difference in size of the two pieces resulting from the cutting. We leave for further research actual computations of the extent to which the subsequent rate ρ can be improved, i.e., by computing the true[21] min-max $\rho^*(S;K;\nu)$. For the a0-governance, one does not have these $K-1$ added constraints on the way to cut the simplex. It is easy to prove in that setup that the ratio $\rho^*(S;\nu)$ cannot be improved for the a0-governance.

When the distribution of shareholders' voting weights is endogeneously determined by the market mechanism from the announced multiplan Y, as for the a1-governance, a result similar to Proposition 3 is more difficult to obtain. One has to prove the existence of a PE $(Y; 1_K; \theta)$ such that, for all k,

1. $\langle Y\rangle$ contains, for all k, the center of gravity of the 'equilibrium' distribution $f \cdot \theta_k$;

2. $f \cdot \theta_k$ ν-concave for some ν.

The following multivariate mean shareholder theorem can be proposed.

Theorem 3 If the distribution $f \cdot \sum_k \theta_k^0$ is ν-concave, then for $-1/S \leq \nu \leq 1$, there exist $\rho^*(S;\nu)$–MSPEa1.

[20] The mean shareholder's type is the one that lies at the center of gravity of the distribution; it is defined as: $\pi_g = (\pi_g^1, \ldots, \pi_g^S)$ with for all s: $\pi_g^s = \int_{\Delta_S} f(\pi)\, \pi^s \, d\pi$

[21] For $\nu = \infty$, $\rho^*(S;K;\infty) = \rho_{S;K}$ as defined by (4).

Proof: See Appendix B. In fact the proof shows that there are, generically with respect to $\{\cdot \sum_k \theta^0_k\}$, up to $\begin{pmatrix} S-1 \\ K-1 \end{pmatrix}$ different subspaces $\langle Y \rangle$ for which the theorem holds. $\square$

This last result sheds some light on the debate on which objective function the firm should optimize in the context of incomplete markets. Firm should make choices that are supported by shareholders. In the present setup, a shareholder is basically characterized by its type π which can be identified as his ideal security. For example Theorem 3 shows existence of production equilibria which are stable for 'acceptable' rates of super majority; they are such that, for firm k, the shareholder whose type, $\pi_{g;k}$, is at the center of gravity of the equilibrium distribution of shares (i.e., the above-mentioned mean shareholder) can exactly span its type, and generate its ideal security: $[\sum_\ell \theta^0(\pi_{g;k})] \cdot \pi_{g;k}$; he could not do better if markets were complete. But to span his ideal security he needs, in general, to buy all securities. (The same line of reasoning holds for the a_0-governance through Proposition 3.)

This result has no direct link with the Drèze criterion. Indeed, the announced security/production plan, y_k, of firm k does not have to be the ideal security/production plan, $\pi_{g;k}$, of this mean shareholder (in general, the multiplan Y, with $y_k = \pi^0_{g;k}$ cannot be supported as a Pareto PE). But the multiplan Y should be such that it contains the ideal security of all mean shareholders; in some way the multiplan Y is optimal for the K mean shareholders. Then the production equilibria are stable for the lowest possible rates of super majority. Finally, it is also the case here that the assumptions securing the result are weaker for the governances à la Drèze, i.e., based on post-trade shares.

4.4 Extensions

Immediate extension to a broader class of utility functions

As easily seen from the proof, the existence of s-MSPE (for governances based on pre-trade shares) in the case $K = S - 1$ (or, trivially, $K = S$) are still valid under assumption (A) and the assumption that $\pi = \mathrm{Argmax}\{U_\pi(x) \mid x \in \Delta_S\}$. Indeed, given a PE $(Y; 1_K; \theta)$, the first order conditions of agents' maximization programs give, for all $k \geq 2$, all π

$$DU_\pi[x(\pi)] \cdot (y_k - y_1) = 0: \tag{8}$$

Suppose $K = S - 1$, when asked whether they agree with an infinitesimal change $u \in \mathbb{R}^S$ (which can be taken orthogonal to $\langle Y \rangle$, given equations (8)) in the production plan of

¯rm k, a shareholder ¼ is indi®erent if and only if ¼ 2 h¹Y i. And the proof of Theorem 2 goes through.

The assumption that ¼ = Argmax f U¼(x) j x 2 ¢ S g can in fact be released. Given a utility function U¼, one can construct the mapping from ¢ S into itself which, to each type ¼ associates its most prefered production plan y¤(¼) = Argmax f U¼(x) j x 2 ¢ S g. The proof of Theorem 2 remains valid under replacement of f by f ± y¤. Therefore the following claim.

Claim 1 Theorem 2 remains valid under assumption (A).

In addition, we can, for symmetric distributions of characteristics (and in particular for the canonical case), extend the result of the paper to the broader class of separable utility functions of the form (3).

Claim 2 Proposition 2 and Theorem 1 remain valid under assumption (A) if in addition the utility functions are taken to be of the separable form (3) and, for Theorem 1, under replacement of f by[22] f ± y¤.

Proof: For Proposition 2, given the symmetry of the distribution of characteristics and identity of the utility functions, ¼ and ¼' will always counter-balance each other in any collective decision making under both governances when departing from the PE (Y ; 1K ; ¹) such that hY i ´ h¹Y i, where ¹Y is de¯ned through equations (5). For Theorem 1, see Appendix C. 2

4.5 Concluding Comments

In fact, it should be possible to extend the analysis to the broader set of assumptions dealt with in Caplin and Nalebu® (1991): production plans are to be taken in an n-dimensional Euclidian space Y, preferences vary accross society and are characterized by a vector ¼ 2 C ½ R S¡1, an (S ¡ 1)-dimensional index of types. The preferences of an agent of type ¼ over the set of allocations x 2 Y (x is a linear combination of the proposed production multiplan) are represented by a continuously di®erentiable utility function U(¼; x). The distribution of types accross society is represented by a probability measure with density

[22] For instance, consider utility functions of the separable form (3) with vs(x) = x¹ with 0 < ¹ < 1. U¼ satisfy assumption (A) and for all ¼ y¤(¼) = [(¼s)1=1¡¹ = P s(¼s)1=1¡¹]S s=1 (notice that ¹ ¡! 0 allows to get the log-linear case). Take then, for the distribution of types over ¢ S, the density f (¼) = [P s(¼s)1=1¡¹]1¡¹ P s(¼s)1¡¹, in which case f ± y¤ is uniform.

f de¯ned over the set of types C. Preferences should satisfy assumptions (A) and (E), or any other which would secure that the types indi®erent to any in¯nitesimal change in the production plan ly on a hyperplane in the space of utility parameters. Moreover the support C of the agents' types should be convex and the distribution f should be º-concave over C.

Even though it is probably within reach to extend the results of Section 4 to this more general setup, and prove that Caplin and Nalebu®'s bound ½(S; º) holds for production equilibria, it is certainly much more di±cult to compute (as in the canonical case) by how much ½(S; K; º) can be improved depending on the number of ¯rms K. It is probably also di±cult to extend the results of Section 4.3 to the case of separable utility functions of the form (3).

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Appendix A : Comments

A family of increasingly homogeneous distributions

In this section we reproduce the preceding analysis for a family[23] of distributions of agents' shares on Δ_S which exhibit an increasing degree of homogeneity. Homogeneity means that agents' beliefs are more concentrated around a particular value of π which is taken here to be the equiprobable belief at the center of Δ_S. For increasing integers c, define

$$d_c(\pi) = (cS-1)!\prod_{s=1}^{S}\frac{(\pi_s)^{c-1}}{(c-1)!}.$$

Figure 2 illustrate the shape of these densities on Δ_2.

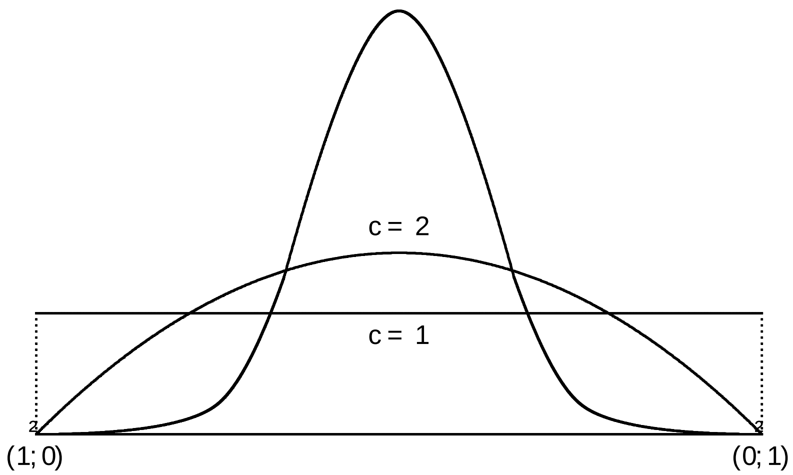

Figure 2

These density functions are clearly log-concave, and therefore they fall into Caplin and Nalebuff (1991) class of conditions guaranteeing an upper bound of $1-1/e$ for the super-majority rules neccessary to ensure stability of the PE.

The same assumptions on the distributions of characteristics are made than in the canonical case, except that the uniform distribution is replced by the distribution defined by the density d_c. This section shows that the needed rates of super majority are smaller ththe bigger c, an intuitive result indeed, but to which exact measure are given. Define, for all S, K and c:

$$\rho^c_{S,K} = 1-\left(\frac{n}{n+1}\right)^{cn}\sum_{j=0}^{c-1}\binom{nc-1+j}{j}\left(\frac{1}{n+1}\right)^{j};$$

with $n = \left\lfloor\frac{S-1}{K}\right\rfloor$.

[23] Crès and Tvede (1998) gives an urn-based model of how these distributions are generated and provides an interpretation in terms of beliefs' formation in the society.

Proposition 4 Fix S, K and c. Then there exist $\rho_{S,K}$–MSPE for all four governances.

For instance, for $c = 2$ and $n = 2$ (resp. $n = 3$, $n = 4$), a 54%-majority rule (resp. , a 55.6, a 56.4%-majority rule) is enough to ensure existence of a MSPE. Moreover Corollary 1 remains valid.

Corollary 2 For any finite c, and $K \geq S/2$, s-MSPE exist for all four governances.

Corollary 3 Fix c. Then for all S, K, there exist ρ–MSPE for all four governances with:

$$\rho = 1 - \left(\sum_{i=0}^{c-1} \frac{c^i}{i!}\right) e^{-c}.$$

These rates are smaller than $\bar{\rho} = 1 - 1/e \approx 0.632$:

$\rho_2 \approx 0.594$	$\rho_4 \approx 0.566$	$\rho_{20} \approx 0.530$
$\rho_3 \approx 0.577$	$\rho_{10} \approx 0.542$	$\rho_{50} \approx 0.519$

and moreover ρ coverges toward $1/2$ when c tends toward infinity. E.g., whatever the degree of market incompleteness, and even when $K = 1$, a 60%-majority rule (resp. 52%-majority rule) is enough to ensure existence of a MSPE for both governances, when $c = 2$ (resp. $c = 50$). All proofs are given in Appendix C.

Robustness to pessimism

We now consider, for $t \in \mathbb{R}$, the utility functions:

$$U_{\pi,t}(x) = \left[\sum_{s=1}^{S} \pi^s \left(\frac{x^s}{\pi^s}\right)^t\right]^{1/t}. \tag{9}$$

They all admit π as their most prefered alternative in Δ_S ($\pi = \operatorname{Argmax}\{U_{\pi,t}(x)\ ; x \in \Delta_S\}$). For $t \to 0$, they define, up to a constant, the log-linear utility functions of the type (3) with for all s, $v^s \equiv \ln$ (log-linear utility functions of this type define in Δ_S, for a fixed π a family of concave indifference curves, `centered' on $\pi$ — indeed, $\pi = \operatorname{Argmax}\{\sum_{s=1}^{S} \pi^s \log x^s\ ; x \in \Delta_S\}$ — and `converging' toward the boundary of Δ_S for utility levels tending toward $-\infty$ (see Figure 3.a)). For negative decreasing values of t, they represent the preferences of an agent always having the prior π on the realization of the states of nature, but who becomes more and more `pessimistic', hence defining a family of indifference surfaces `centered' on π but which are more and more triangular (in the case $S = 3$, see Figure 3.b). And for $t = -\infty$, $U_{\pi,t}$ becomes:

$$U_{\pi,-\infty}(x) = \min_s \left\{\frac{x^s}{\pi^s}\right\}; \tag{10}$$

whose indifference surfaces in Δ_S are simplices of dimension $S - 1$ (represented in the case $S = 3$ on Figure 3.c). This last case holds for agents which are completely pessimistic: they only care for lowest consumption in period 1 (relative to the probability of realization of the state of nature).

Assume that agents are characterized by the utility functions $U_{\pi,-\infty}$. Assume moreover that they are symmetrically distributed over Δ_S and allocated symmetric initial shares of the firms. Suppose finally that for all k, the total quantity of shares distributed is constant accross firms: $\Gamma^0_k = \Gamma^0$ is independent of k. Then we have the following proposition.

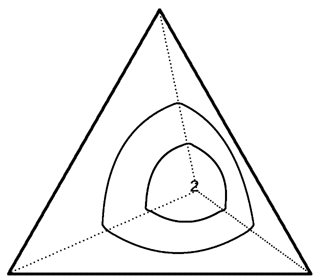

Figure 3.a

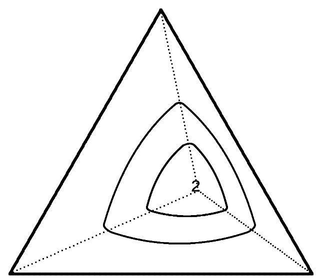

Figure 3.b

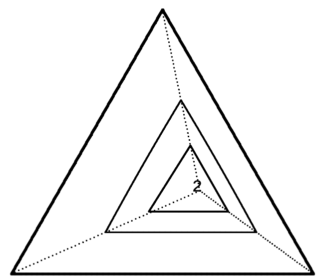

Figure 3.c

Proposition 5 *The min-max majority, for all governances, is*

$$\rho_{S,K} = \frac{\binom{S-1}{K-1}}{\binom{S-1}{K}+1}.$$

Proof: See Appendix C. □

We can draw two remarks from this last result.

1. This entails that in Caplin and Nalebuff (1988, 1991) case where $K = 1$, the min-max majority is $\rho_{S,1} = \frac{S-1}{S}$. Hence for a big number of states of the world, only rates close to unaninity secure that the center of gravity of the support of agents' types is majority stable. Greenberg (1979) proves that for decision problems in R^{S-1} with $K = 1$ and individual with convex preferences, the min-max is bounded above by $\frac{S-1}{S}$. For those pessimistic preferences, Greenberg (1979)'s bound of $\frac{S-1}{S}$ is then reached.
2. The benefit, in terms of majority stability, of having access to many assets in this case is much bigger than in the case of separable preferences: if $n = \binom{S-1}{K}$ is not too big, $\rho_{S,K}$ keeps tractable values (e.g., 67% if $n = 3$), whereas it is close to unaninity when K is small. In particular, for any finite S, K, it is still the case that s–MSPE exist as soon as $K \geq S/2$ for all governances.

Appendix B : Proofs of Sections 2, 3 and 4

Proof of Lemma 1: Consider a production multiplan $Y = (y_k)_{k=1}^{K}$, generating a $(K-1)$-dimensional subspace $\langle Y \rangle$. Define, for agent π the normalized optimal portfolio $\theta(\pi)$ which solves the following program:

$$\max_{\theta(\pi)} \quad \bar{U}_{\pi,Y}[\theta(\pi)]$$
$$\text{s. t.} \quad \sum_{k=1}^{K} \theta_k(\pi) = 1$$

The resulting normalized optimal consumption $\bar{x}(\pi) = \sum_{k=1}^{K} \theta_k(\pi)\, y_k$ is then the point where the indifference surface of agent π in Δ_S is tangent to $\langle Y \rangle$. One then has (dropping π): $\overrightarrow{y_1\bar{x}} = \sum_{k=2}^{K} \theta_k \overrightarrow{y_1 y_k}$.

Define the multiplan $\tilde{Y} = (\tilde{y}_k)_{k=1}^{K}$ such that $\tilde{y}_1 = y_1$ and,

$$\text{for } k \geq 2; \quad \overrightarrow{y_1\tilde{y}_k} = \alpha_k \, \overrightarrow{y_1\bar{y}_k} \quad \text{with} \quad \alpha_k = \frac{\displaystyle\int_{\Delta_S} f(\pi) \left[\sum_k \theta_k^0(\pi) \quad \bar{\theta}_k(\pi)\right] d\pi}{\displaystyle\int_{\Delta_S} f(\pi)\, \theta_k^0(\pi)\, d\pi} : \tag{11}$$

Define moreover, for $k \geq 2$,

$$\tilde{\theta}_k(\pi) = \left[\sum_k \theta_k^0(\pi)\right] \frac{\displaystyle\int_{\Delta_S} f(\pi)\, \theta_k^0(\pi)\, d\pi}{\displaystyle\int_{\Delta_S} f(\pi) \left[\sum_k \theta_k^0(\pi) \quad \bar{\theta}_k(\pi)\right] d\pi} \, \bar{\theta}_k(\pi) \quad \text{and} \quad \tilde{\theta}_1(\pi) = \sum_k \theta_k^0(\pi) - \sum_{k \geq 2} \tilde{\theta}_k(\pi) :$$

One then obtains, for all k:

$$\int_{\Delta_S} f(\pi)\, \tilde{\theta}_k(\pi)\, d\pi = \int_{\Delta_S} f(\pi)\, \theta_k^0(\pi)\, d\pi;$$

and moreover $\tilde{\theta}(\pi) = \left[\tilde{\theta}_k(\pi)\right]_{k=1}^{K}$ obviously solves the program:

$$\begin{array}{ll} \max\limits_{\theta(\pi)} & U_{\pi,\tilde{Y}}[\theta(\pi)] \\ \text{s. t.} & \displaystyle\sum_{k=1}^{K} \theta_k(\pi) = \sum_{k=1}^{K} \theta_k^0(\pi) \end{array}$$

Therefore $(\tilde{Y}; 1_K; [\tilde{\theta}(\pi)]_{\pi \in \Delta_S})$ is a PE. $\square$

Proof of Proposition 1: The proof goes through three steps. The first step computes the optimal consumption $\tilde{x}(\pi)$, and, for the sake of illustration, proves that $(\tilde{Y}; 1_K; \tilde{\theta})$ as defined by (7) is a PE under the assumptions of the example following Proposition 1. The second and third steps prove that any PE $(\tilde{Y}; 1_K; \tilde{\theta})$ such that $\langle\tilde{Y}\rangle \equiv \langle\bar{Y}\rangle$ is a $\rho_{S,K}$–MSPE for all governances: the second step is the construction of the best way to challenge a PE within each firm; the third step is the computation of the corresponding ratios $\rho_{S,K}$.

Step 1: Suppose the K firms announce the production multi-plan $\tilde{Y} = (\tilde{y}_k)_{k=1}^{K}$. Consider the market prices to be 1_K. Then the maximization program of a agent π $M(\pi)$, is equivalent to findng the portfolio $\tilde{\theta}(\pi)$ which maximizes the utility function: $U_{\pi,\tilde{Y}}[\tilde{\theta}(\pi)]$ subject to the constraint $\sum_k \tilde{\theta}(\pi) = 1$. Given assumption (E), and the identity $\tilde{x}(\pi) = \sum_k \tilde{\theta}_k(\pi)\, \tilde{y}_k = \bar{x}(\pi) = \sum_k \bar{\theta}_k(\pi)\, \bar{y}_k$, it is defined by the orthogonal projection, $\bar{x}(\pi) = \sum_k \bar{\theta}(\pi)\bar{y}_k$, of π on $\langle\bar{Y}\rangle$. Hence $[\bar{x}(\pi) - \pi] \cdot [\bar{y}_1 - \bar{y}_k] = 0$ for all $k \geq 2$.

We have[24]: $\bar{x}^s = \bar{\theta}_k/(n+1)$ if $s \in S_k$ and $\bar{x}^s = \bar{\theta}_k/n$ if $s \in T_k$. Therefore, for all $k \geq 2$:

$$\sum_s (\bar{x}^s - \pi^s)(\bar{y}_1^s - \bar{y}_k^s) = 0 \; (\;) \quad \left\{ \begin{array}{lll} \bar{\theta}_k = \pi^{S_k} + (\bar{\theta}_1 - \pi^{S_1}) & \text{if} & k \leq m \\ \bar{\theta}_k = \pi^{T_k} + \dfrac{n}{n+1}(\bar{\theta}_1 - \pi^{S_1}) & \text{if} & k \geq m+1 \end{array} \right. :$$

These equations, along with the condition $\sum_k \bar{\theta}_k = 1$, straightforwardly imply: $\bar{\theta}_k = \pi^{S_k}$ when $k \leq m$ and $\bar{\theta}_k = \pi^{T_k}$ when $k \geq m+1$.

---

[24] In heavy computations, when no confusion can be feared, $\bar{x}$ and $\bar{\theta}$ will stand for $\bar{x}(\pi)$ and $\bar{\theta}(\pi)$.

We then have to check that 1_K is an equilibrium price. We should have $\int_{\Delta_S} f(\pi)\,\tilde\theta_k(\pi)\,d\pi = \Gamma_k^0 = \frac{1}{K}$, for all k. For $k \le m$, a standard application of Fubini's theorem[25] gives:

$$(S-1)!\int_{\Delta_S} \pi^{S_k}\,d\pi = (S-1)!\int_0^1 \pi\frac{\pi^n}{n!}\frac{(1-\pi)^{S-n-2}}{(S-n-2)!}\,d\pi = \frac{n+1}{S};$$

hence the above equilibrium equation is fulfilled. Through an identical line of computation, the reader can check that the equilibrium equations for $k \ge m+1$ is fulfilled.

Step 2: Up to now, we have proved that $\bar E = (\bar Y; 1_K; \tilde\theta)$ is a PE. We have to prove that it is ρ-majority stable for the lowest possible ρ. The first needed lemma here is a replication of Caplin and Nalebuff (1988) Proposition 2.

Lemma 2 For all k, $P_{\bar E;\tilde\theta}(\bar y_k)$ (resp. $A_{\bar E;\tilde\theta}(\bar y_k)$) is the largest fraction of shareholders with positive share (resp. the largest fraction of positive shares) on either side of any hyperplane through $\bar Y$.

Proof: Fix z_k, then from the strict concavity of the agents' preferences, $z_k(\lambda) = \lambda\bar y_k + (1-\lambda)z_k$ will get a larger fraction than z_k of shareholders and shares against $\bar y_k$. Indeed, since $x + \theta[z_k(\lambda) - \bar y_k] = \lambda x + (1-\lambda)[x + \theta(z_k - \bar y_k)]$ then for $\lambda \ne 0, 1$, $U[x + \theta(z_k(\lambda) - y_k)] > \lambda U[x] + (1-\lambda)U[x + \theta(z_k - \bar y_k)]$; hence $U[x + \theta(z_k - \bar y_k)] \ge U[x]$ entails $U[x + \theta(z_k(\lambda) - \bar y_k)] \ge U[x]$.

Therefore, in looking for $P_{\bar E;\tilde\theta}(\bar y_k)$ or $A_{\bar E;\tilde\theta}(\bar y_k)$ we can focus on alternative production plan z_k arbitrarily close to $\bar y_k$. We know that at a PE, the optimal portfolio is such that the indifference surface of each agent is tangent to $\langle\bar Y\rangle$ at the equilibrium consumption. Therefore in the limit, shareholders whose welfare is improved by the infinitesimal move from $\bar y_k$ are separated from those whose welfare is impaired by a hyperplane (thanks to assumption (E)) containing $\langle\bar Y\rangle$ with normal, given equations (8), the orthogonal projection of $z_k - \bar y_k$ on the orthogonal of $\langle\bar Y\rangle$. Hence z_k can be restricted to converge toward $\bar y_k$ orthogonally to $\langle\bar Y\rangle$ and in the limit, shareholders are separated by an hyperplane containing $\langle\bar Y\rangle$ with normal $z_k - \bar y_k$.

The general equation of an hyperplane containing $\langle\bar Y\rangle$ is (the intersection with Δ_S of):

$$\sum_{k=1}^{K}\sum_{s\in S_k(T_k)}\alpha^s\pi^s = 1; \quad \text{with} \quad \begin{cases} \forall k \le m; & \sum_{s\in S_k}\alpha^s = n+1 \\ \forall k \ge m+1; & \sum_{s\in T_k}\alpha^s = n \end{cases}; \tag{12}$$

that one denotes $\langle\alpha\rangle$, where $\alpha = (\alpha^s)_{s=1}^S$. One can easily check that the normal vector, in Δ_S, of $\langle\alpha\rangle$ is $\alpha - 1_S$. $\square$

A second lemma is needed.

Lemma 3 For all k, among all hyperplanes through $\bar Y$, the ones that divide Δ_S into two pieces with most unequal volumes (whatever the governance) are those defined by equation (12) with for for some $k \le m$, $\alpha^s = \frac{n+1}{n}$ for $s \in S_k$, and for all $s \in (S\setminus S_k) \cup T$, $\alpha^s = 1$.

[25] The volume computed throughout the paper are those of the projection of Δ_S on the last variable, i.e., of the set of vector $\lambda \in \mathbb{R}_+^{S-1}$ such that $\sum_{s=1}^{S-1}\lambda^s \le 1$.

Proof: Let us prove the lemma in the most difficult case, i.e., for the a1-governance, the other cases are a straightforward simplification of the following proof. Define the projection from the $(S-1)$–dimensional simplex onto the $(K-1)$–dimensional simplex:

$$
\begin{array}{cccc}
P & : & \Delta_S & \longrightarrow & \Delta_K \\
 & & (\pi^1, \ldots, \pi^S) & \longrightarrow & (\pi^{S_1}, \ldots, \pi^{S_m}, \pi^{T_{m+1}}, \ldots, \pi^{T_K})
\end{array}
$$

Fix a vector of shares $(\mu_k)_{k=1}^K$ (and thus $(\bar\mu_k)_{k=1}^K$) in Δ_K. The set of agents having exactly those shares are in $P^{-1}[(\mu_k)]$. We are now reduced to the problem of dividing $P^{-1}[(\mu_k)]$ as unevenly as possible by an hyperplane going through its center point. But $P^{-1}[(\mu_k)]$ is a cartesian product of simplices of dimension either n (for $k \le m$) or $n-1$ (for $k \ge m+1$), with equation: $\sum_{s \in S_k(T_k)} \pi^s = \mu_k$. This very structure will make the former task simple: it is sufficient to divide one of the base-simplex (e.g., k) as unevenly as possible (measure-wise) and consider the cartesian product of the big portion with all other simplices. From Caplin and Nalebuff (1988) we know that a base-simplex have to be divided by an hyperplane parallel to one face: among all hyperplanes of the form $\sum_{s \in S_k(T_k)} \alpha^s \pi^s = 1$ we have to choose one such that $\alpha^s_k = \frac{n+1}{n}$ for all but one s in S_k (or $= \frac{n}{n-1}$ for all but one s in T_k); the portion between this hyperplane and the chosen face is the biggest possible.

It can finally be easily checked that this amounts to propose an infinitesimal change of the production plan y_k which sacrifices one state of nature to the benefit of all others and implement for example the change: $\left(-\varepsilon, \frac{\varepsilon}{S-1}, \ldots, \frac{\varepsilon}{S-1}\right)$. The same cutting occurs if the infinitesimal change of the payoffs of the states in S_1 only: $\left(-\varepsilon, \frac{\varepsilon}{n}, \ldots, \frac{\varepsilon}{n}, 0, \ldots, 0\right)$ (orthogonal to $\langle\pi\rangle$) is proposed. $\square$

Step 3: Let us then compute the proportion of shareholders on the smallest side of the hyperplane defined by equation (12) with, according to Lemma 3 (let us do it for firm 1 without loss of generality): $\alpha^1 = 0$, $\alpha^s = \frac{n+1}{n}$ for $s \in S_1 \backslash 1$ and $\alpha^s = 1$ for $s \in (S \backslash S_1) \cup T$. We thus compute the relative volume, in Δ_S, of the volume defined by the equation:

$$
\frac{n+1}{n} \pi^{S_1 \backslash 1} + \pi^{(S \backslash S_1) \cup T} \le 1 : \tag{13}
$$

The proportion, in Δ_S, of this last volume is (apply Fubini's theorem), denoting $u = \pi^{S_1 \backslash 1}$ and $v = \pi^{(S \backslash S_1) \cup T}$:

$$
(S-1)! \int_{u=0}^{\frac{n}{n+1}} \frac{u^{n-1}}{(n-1)!} \left(\int_{v=0}^{1-\frac{n+1}{n}u} \frac{v^{S-n-2}}{(S-n-2)!} \, dv \right) du ;
$$

which is equal to $\left(\frac{n}{n+1}\right)^n$. This last ratio is $1 - \rho_{S;K}$. It is a minimal ratio of shareholders that will oppose to any change of production plan within each firm k, since at the equilibrium, everybody is allocated positive shares of the firms so that everybody gets the right to vote.

This proves the Proposition for the p0, p1 and a0-governances.

For the a1-governance, the argument developped in Lemma 3 still holds to characterize the best way to challenge the status quo within a firm. Therefore, for $k \le m$, one has to compute (for firm 1, the same result will hold for other firms) the measure of the volume defined by inequality (13), weighted by $\pi^{S_1} = 1 - v$ (ignoring the constant term $\frac{S}{(n+1)K}$ which will disappear in the normalization). It is

$$
(S-1)! \int_{u=0}^{\frac{n}{n+1}} \frac{u^{n-1}}{(n-1)!} \left(\int_{v=0}^{1-\frac{n+1}{n}u} \frac{v^{S-n-2}}{(S-n-2)!} (1-v) \, dv \right) du ;
$$

which is, after standard computations, equal to $\frac{n+1}{S}\left(\frac{n}{n+1}\right)^n$. (The same result holds true if one weights by π^{S_k} instead of π^{S_1}.) This has to be normalized by the volume of Δ_S weighted by $\pi^{S_1} = u'$, which gives

$$\int_{u'=0}^{1} u' \frac{(u')^n}{n!} \frac{(1-u')^{S-n-2}}{(S-n-2)!} du' = \frac{n+1}{S}.$$

Therefore the relative volume we are looking for is the same as in the one person-one vote governance.

For all $k \geq m+1$, the volume defined by inequality (13), weighted by π^{T_k} is equal to $\frac{n}{S}\left(\frac{n}{n+1}\right)^n$. Therefore, if one takes the density $\tilde{\mu}_k = \pi^{T_k} + \frac{1}{(n+1)K}\pi^S$ one gets $\frac{1}{S}\left(n+\frac{m}{K}\right)\left(\frac{n}{n+1}\right)^n$. This has to be normalized by the volume of Δ_S weighted by the same density, which is $\frac{1}{S}\left(n+\frac{m}{K}\right)$. $\square$

Proof of Theorem 2: For the existence of s–MSPEa1, we consider the same family of hyperplanes $\langle\varphi\rangle_{\varphi\in S_{S-2}}$ as the one defined for the a0-governance. For each φ, fix a orthonormed basis $(e^\ell_\varphi)_{\ell=1}^{S-2}$ which moves continuously with φ. Therefore $e^\ell_{-\varphi} = -e^\ell_\varphi$, for all φ, all ℓ. Denote C_φ the subset of $\langle\varphi\rangle$ of optimal consumptions $x_\varphi(\pi)$ for all types π (for a given multiplan Y_φ continuously depending on φ such that $[Y_\varphi, 1_K, \mu_\varphi]$ be a PE satisfying the Pareto criterion, and of course $\langle Y_\varphi\rangle = \langle\varphi\rangle$). One has for $x_\varphi \in C_\varphi$, $x_\varphi = \sum_\ell x^\ell_\varphi e^\ell_\varphi$.

For $x_\varphi \in C_\varphi$, let $f\Theta^{0+}_\varphi(x_\varphi)$ [resp. $f\Theta^{0-}_\varphi(x_\varphi)$] denote the density obtained from $f\Delta\sum_k \mu^0_k$ by aggregating all types $\pi \in \langle\varphi\rangle^+$ [resp. $\pi \in \langle\varphi\rangle^-$] such that $x_\varphi(\pi) = x_\varphi$. One has for all k, $1 \leq k \leq K-1$,

$$\int_{\langle\varphi\rangle^*} f(\pi)\mu_k(\pi)d\pi = \int_{C_\varphi} f\Theta^{0*}_\varphi(x_\varphi)\tilde{\mu}_{k,\varphi}(x_\varphi)dx_\varphi , \tag{14}$$

for $* = +$ or $-$, and where $\tilde{\mu}_{k,\varphi}(x_\varphi) = \mu_k(\pi) = \sum_\ell \mu^0(\pi)$ for all π such that $x_\varphi(\pi) = x_\varphi$.

Consider the $S-2$ mappings $(\mu_\ell)_{\ell=1}^{S-2}$ on S_{S-2} defined by

$$\mu_\ell(\varphi) = \int_{C_\varphi} x^\ell_\varphi[f\Theta^{0+}_\varphi - f\Theta^{0-}_\varphi](x_\varphi)\, dx_\varphi . \tag{15}$$

They all satisfy the symmetry property: $\mu_\ell(-\varphi) = -\mu_\ell(\varphi)$. Indeed, given that $e^\ell_{-\varphi} = -e^\ell_\varphi$, for all φ $x_{-\varphi} = \alpha_\varphi - x_\varphi$ for some fixed α_φ, from which we get, trhough an obvious change of coordinates:

$$\mu_\ell(-\varphi) = \int_{C_\varphi} (\alpha^\ell_\varphi - x^\ell_\varphi)[f\Theta^{0+}_{-\varphi} - f\Theta^{0-}_{-\varphi}](\alpha_\varphi - x_\varphi)[-dx_\varphi] = \int_{C_\varphi} (\alpha^\ell_\varphi - x^\ell_\varphi)[f\Theta^{0+}_\varphi - f\Theta^{0-}_\varphi](x_\varphi)dx_\varphi$$

because obviously $f\Theta^{0+}_{-\varphi}(x_{-\varphi}) = f\Theta^{0-}_\varphi(x_\varphi)$. And we know that for all φ, $\int_{C_\varphi} \alpha^\ell_\varphi[f\Theta^{0+}_\varphi - f\Theta^{0-}_\varphi](x_\varphi)dx_\varphi = 0$ since $\langle\varphi\rangle$ divides Δ_S into two pieces of equal measure with respect to the distribution $f\Delta\Theta^0$.

Given that mappings $(\mu_\ell)_{\ell=1}^{S-2}$ satisfy the above described symmetry property, we get, by an indirect corollary of the Borsuk-Ulam theorem, that they possess a common zero. Denote it φ_0. Hence for all ℓ, $1 \leq \ell \leq S-2$, one has:

$$\int_{C(\varphi_0)} x^\ell_{\varphi_0} f\Theta^{0+}_{\varphi_0}(x_{\varphi_0})\, dx_{\varphi_0} = \int_{C(\varphi_0)} x^\ell_{\varphi_0} f\Theta^{0-}_{\varphi_0}(x_{\varphi_0})\, dx_{\varphi_0} .$$

Lastly, one observes (as shown in the proof of Theorem 3) that for all k, $1 \leq k \leq K-1$, $\tilde{\mu}_k(x_{\varphi_0})$ is an affine function of x_{φ_0}. Therefore, from equation (14), we get the result: φ_0 defines a PE $[Y_{\varphi_0}, 1_K, \mu_{\varphi_0}]$

that divides Δ_{S} into two pieces of equal measure with respect to the distributions $f_\Delta\theta_{k;\varphi_0}$, for all k, $1 \leq k \leq K-1$, and for $f_\Delta \sum_k \theta_{k;\varphi_0}$, hence also for $f_\Delta\theta_{K;\varphi_0}$. $\Box$

Proof of Theorem 3: The proof goes through three steps. The first step to prove that the relevant PE $(Y; 1_K; \Theta)$ should generate a subspace $\langle Y\rangle$ that contains the center of gravity, $\pi_g = (\pi^s_g)_{s=1}^S$, of the distribution $f_\Delta \sum_k \theta^0_k$ (denoted f_{Θ^0} in the sequel). The second step to show that among all multiplans Y such that $\langle Y\rangle$ contains π_g, there exists one that contains the center of gravity, $\pi_{g;k}$, of the distribution $f_\Delta\theta_k$, for all k. The third step shows that the density $f_\Delta\theta_k$ is γ-concave for all k.

Step 1: The following statement is proved here: Consider a PE $(Y; 1_K; \Theta)$ such that, for all k, $\langle Y\rangle$ contains $\pi_{g;k}$, then it contains π_g.

Indeed, by definition:

$$\forall s;\ \pi^s_{g;k} = \frac{1}{\Gamma^0_k}\int_{\Delta_S} \pi^s f(\pi)\,\theta_k(\pi)\,d\pi;$$

and one can easily check that π_g is the barycenter of $(\pi_{g;k})_{k=1}^K$ with weights $\left(\frac{\Gamma^0_k}{\Gamma^0}\right)_{k=1}^K$, with $\Gamma^0 = \sum_{k=1}^K \Gamma^0_k$, i.e., $\pi_g = \frac{1}{\Gamma^0}\sum_{k=1}^K \Gamma^0_k\,\pi_{g;k}$. Therefore, a relevant multiplan Y is such that $\pi_g \in \langle Y\rangle$.

Step 2: We show now that among all multiplans Y such that $\langle Y\rangle$ contains π_g, there exist at least one that contains $\pi_{g;k}$, for all k. This will go through two steps. The first step to prove that there exists $S-1$ independent stable directions in $\langle\Delta_S\rangle$; the definition of a stable direction v follows: consider the distribution $x_\Delta f_{\Theta^0}$ over Δ_S where $x(\pi)$ is the abscissa of the orthogonal projection of π on the direction of v, denote $\pi_g(v)$ the center of gravity of this distribution, v is stable if the vector $\overrightarrow{\pi_g\pi_g(v)}$ is colinear to v. The second step shows that any subspace containing $K-1$ of these independent stable directions can be spanned by a K-multiplan Y such that $\langle Y\rangle$ contains $\pi_{g;k}$, for all k.

Step 2.1: Stable directions. Define, for all state of nature t, the vector $u(t) \in \langle\Delta_S\rangle$, with $u^t(t) = -1$ and $u^s(t) = 1/(S-1)$ for $s \neq t$. The orthogonal projection of Δ_S on the direction defined by $u(t)$ is only function of π^t, i.e., all $\pi \in \Delta_S$ with same t-th coordinate project on the same point on any line spanned by $u(t)$. Denote $\pi_g(t) = [\pi^s_g(t)]_{s=1}^S$ the center of gravity of the distribution $\pi^t f_{\Theta^0}(\pi)$; by definition:

$$\forall s;\ \pi^s_g(t) = \frac{1}{\Gamma^0\pi^t_g}\int_{\Delta_S} \pi^s\,\pi^t\,f_{\Theta^0}(\pi)\,d\pi.$$

One immediatly gets the following property:

$$\sum_{t=1}^S \pi^t_g\,\pi_g(t) = \pi_g;\ \text{i.e.};\ \pi_g \text{ is the barycenter of the family } [\pi_g(t)]_{t=1}^S \text{ with weights } (\pi^t_g)_{t=1}^S. \quad (16)$$

The independent family of vectors $[u(t)]_{t=1}^{S-1}$ spans $\langle\Delta_S\rangle$. Consider a vector $v = \sum_{t=1}^{S-1} v^t\,u(t)$, and denote $\langle v\rangle$ the direction it spans in $\langle\Delta_S\rangle$. We have the following property:

$$\forall(\pi, \pi');\ \text{if}\ \sum_{t=1}^{S-1} v^t\pi^t = \sum_{t=1}^{S-1} v^t\pi'^t\ \text{then}\ \pi, \pi' \text{ project orthogonally on the same point on } \langle v\rangle. \quad (17)$$

Indeed, one then has: $\overrightarrow{\pi\pi^0}\cdot v = \sum_{t=1}^{S-1} v^t\, u(t)\cdot\overrightarrow{\pi\pi^0}$, with $u(t)\cdot\overrightarrow{\pi\pi^0} = -\frac{S}{S-1}(\pi^0 - \pi^t)$; therefore $\overrightarrow{\pi\pi^0}\cdot v = -\frac{S}{S-1}\sum_{t=1}^{S-1} v^t(\pi^0 - \pi^t) = 0.$

Consider the linear function $x(\pi) = \sum_{t=1}^{S-1} v^t\pi^t$. Property (17) states that the orthogonal projection of π on $\langle v\rangle$ is only a function of $x(\pi)$. Denote $\pi_g(v)$ the center of gravity of the distribution $x\cdot f\Gamma^0$; standard computations give:

$$\pi_g(v) = \sum_{t<S}\left(\frac{v^t\pi_g^t}{\sum_{s<S} v^s\pi_g^s}\right)\pi_g(t).$$

We are now looking for the vectors v such that $\overrightarrow{\pi_g\pi_g(v)}$ is colinear to v. From the last equation we get that $\overrightarrow{\pi_g\pi_g(v)}$ is colinear (for a v in general position) to the vector $\left[\sum_{t<S} v^t\pi_g^t(\pi_g^s(t) - \pi_g^s)\right]_{s=1}^{S}$.

Denote $a^{st} = \pi_g^t(\pi_g^s(t) - \pi_g^s)$. Then $\overrightarrow{\pi_g\pi_g(v)}$ is colinear to v, if and only if there exists $\lambda \neq 0$ such that for all s,

$$\sum_{t<S} a^{st}v^t = \lambda\sum_{t<S} u^s(t)v^t\,.$$

Obviously, if the latter equation is satisfied for $s = 1, \ldots, S-1$, it is also satisfied for $s = S$. Therefore, denoting A, resp. U, the $(S-1)$–square matrix $(a^{st})_{s,t=1}^{S-1}$, resp. $(u^s(t))_{s,t=1}^{S-1}$, denoting w be the vector with coordinates $(v^t)_{t=1}^{S-1}$ in the original basis, the following property holds:

$$\overrightarrow{\pi_g\pi_g(v)}\ \text{ is colinear to } v \text{ if and only if } Aw \text{ is colinear to } Uw\,. \tag{18}$$

One has to look for the eigenvectors (and eigenvalues) of the product matrix: $(-U)^{-1}A$, and prove that they are all real and that $(-U)^{-1}A$ is diagonalizable. The two matrices A and $-U$ have the following properties:

- A is real symmetric (therefore Hermitian) and for all s, $\sum_{t<S} a^{st} > 0$. (Indeed, by property (16) $\sum_{t\leq S} a^{st} = 0$ and obviously $a^{ss} > 0$, $a^{st} < 0$ for $s \neq t$.) In fact, it is a variance-covariance matrix.
- $-U$ is real symmetric; its inverse is:

$$(-U)^{-1} = \frac{S-1}{S}\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix};$$

  which is obviously a positive definite Hermitian matrix.

And one knows that the product of a positive definite matrix and a Hermitian matrix is a diagonalizable matrix, all of whose eigenvalues are real (see, e.g., Horn and Johnson (1985), Theorem page 465).

As a remark, one can easily check that if the distribution $f\cdot\sum_k \theta_k^0$ is uniform, $A = -\frac{S-1}{S^2(S+1)}U$ therefore all directions are stable: whatever the subspace going through π_g, it contains $\pi_{g,k}$ for all k.

Step 2.2: Fix $K \le S$. Pick $K-1$ of the preceding stable directions: $V = (v_j)_{k=1}^{K-1}$, with for all j: $v_j = \sum_{t=1}^{S-1} v_j^t\, u(t)$, and denote $\langle V\rangle$ the subspace they span in $\langle\Delta_S\rangle$. Property (17) generalizes:

$$\forall(\pi,\pi');\ \text{ if } \forall j \sum_{t=1}^{S-1} v_j^t \pi^t = \sum_{t=1}^{S-1} v_j^t \pi'^t \text{ then } \pi, \pi' \text{ project orthogonally on the same point in } H(V): \quad (19)$$

Consider for all $j < K-1$, $x_j(\pi) = \sum_{t=1}^{S-1} v_j^t \pi^t$. Property (19) states that the orthogonal projection of π on $\langle V\rangle$ is only a function of $[x_j(\pi)]_{j=1}^{K-1}$. Denote $\pi_g(j)$ the center of gravity of the distribution $x_j(\pi) f\Theta^0(\pi)$. Take a PE $(Y; 1_K; \Theta)$ such that $\langle Y\rangle \equiv \langle V\rangle$. A direct consequence of Property (19) is that for all k, $1 \le k \le K$, $\tilde\theta_k(\pi) = \frac{\mu_k(\pi)}{\sum_\ell \mu^\ell(\pi)}$ will be an affine function of $[x_j(\pi)]_{j=1}^{K-1}$:

$$\tilde\theta_k(\pi) = \alpha_k + \sum_{j=1}^{K-1} \beta_{kj}\, x_j(\pi):$$

Standard computations give:

$$\begin{aligned}
\pi_{g;k} &= \frac{1}{\bar\theta_k^0} \int_{\Delta_S} \pi\tilde\theta_k(\pi)\, f\Theta^0(\pi)\, d\pi \\
&= \frac{\alpha_k}{\alpha_k + \sum_j \beta_{kj}\gamma_j}\, \pi_g + \sum_{j=1}^{K-1} \frac{\beta_{kj}\gamma_j}{\alpha_k + \sum_j \beta_{kj}\gamma_j}\, \pi_g(j);
\end{aligned}$$

where $\gamma_j = \sum_t v_j^t \pi_g^t$. Therefore $\pi_{g;j}$ is in $\langle V\rangle \equiv \langle Y\rangle$ which ends this step.

Step 3: The third step is to show that the density $f\cdot\mu_k$ is γ-concave for all k. This comes immediately from the fact that $f(\pi)\cdot\mu_k(\pi) = f\Theta^0(\pi)\tilde\theta_k(\pi)$ where, as seen before, $\tilde\theta_k(\pi)$ is an affine function of π hence is γ-concave for $\gamma \le 1$. $\square$

Proof of Claim 2: For the sake of lightness of the notation we consider $m = K$. Suppose the K firms announce a production multi-plan $Y = (y_k)_{k=1}^K$, such that $\langle Y\rangle = \langle\tilde Y\rangle$ (as defined by (5)), and such that $(Y; 1_K; \mu)$ is a PE (allowed by Lemma 1). The identity $x(\pi) = \sum_k \mu_k(\pi)\, y_k = \tilde x(\pi) = \sum_k \tilde\theta_k(\pi)\, \tilde y_k$ allows to focus on $(\tilde Y; \tilde\theta)$.

The utility levels of agent π on feasible consumptions is: $\sum_k \pi^{S_k} v^{S_k}[\tilde\theta_k/(n+1)]$ with $v^{S_k} \equiv \sum_{s\in S_k} v^s$. This yields the optimal portfolio $(\tilde\theta_k)_k$ as a function of $(\pi^{S_k})_{k=1}^K$: $(\tilde\theta_k)_k = V[(\pi^{S_k})_k]$, where V is one-to-one. Indeed, for all j, the first order conditions of the maximization program give $\pi_j Dv^{S_j}(\tilde\theta_j) =$ constant; hence, since $Dv^{S_j}(\tilde\theta_j) > 0$, $V[(\pi^{S_k})_k] = V[(\pi'^{S_k})_k] \Rightarrow (\pi^{S_k})_k = (\pi'^{S_k})_k$). As examples, for the log-linear case $v^s = \ln$ for all s, V is the identity: $V[(\pi^{S_k})_k] = (\pi^{S_k})_k$. If $v^s(x) = x^\nu$ with $0 < \nu < 1$, $V[(\pi^{S_k})_k] = \left[\frac{(\pi^{S_k})^{1/1-\nu}}{\sum_k (\pi^{S_k})^{1/1-\nu}}\right]_k$.

Fix a vector of shares $(\tilde\theta_k)_{k=1}^K$ in Δ_K. Then all agents with type in $P^{-1}[V^{-1}(\tilde\theta_k)]$ (where P is defined in Lemma 3) have the same portfolio, hence the same optimal consumption under the assumtion of the canonical case. Moreover $P^{-1}[V^{-1}(\tilde\theta_k)]$ is, as stated in Lemma 3, a cartesian product of simplices of dimension n. Consider the cutting proposed in Lemma 3, through a change $\left(-\epsilon; \frac{\epsilon}{S-1}; \dots; \frac{\epsilon}{S-1}\right)$ in the production plan of a firm. We know it does not divide the parameter space Δ_S through a hyperplane.

But it does divide $P^{-1}[V^{-1}(\mu_k)]$ though a hyperplane, as unevenly as possible with respect to the uniform distribution of most prefered alternative $f_{\delta_{y^*}}$, and the biggest piece gets the same proportion of the $(n/n+1)^n$ of the volume of this base simplex. The distribution $f_{\delta_{y^*}}$ being uniform, this last ratio holds for the whole set Δ_S.

One can finally check (for Proposition 2) that any proposed change (that can be taken orthogonal to $\langle \mathbb{1} \rangle$), will always put π and π' (who always select the same portfolio and optimal consumption) in different positions: if it impairs the welfare of one, it improves the welfare of the other. $\square$

Appendix C : Proofs of Appendix A

Proof of Proposition 4: The first step of the proof of Proposition 1 still goes through. The only difference is in the equilibrium equation for prices which should be computed for the new density d_c in the case of the example based on the multiplan Υ defined by equations (7). It is $\int_{\Delta_S} \mu_k(\pi)\, f(\pi)\, d\pi = (cS-1)! \int_{\Delta_S} \frac{1}{K}\, d_c(\pi)\, d\pi$ for all k. For $k \leq m$, one gets:

$$\int_{\Delta_S} \pi_{S_k}\, d_c(\pi)\, d\pi = \int_0^1 \pi \frac{\pi^{c(n+1)-1}}{[c(n+1)-1]!} \frac{(1-\pi)^{c(S-n-1)-1}}{[c(S-n-1)-1]!}\, d\pi = \frac{c(n+1)}{cS} = \frac{(n+1)}{S}\,;$$

and of course $\int_{\Delta_S} d_c(\pi)\, d\pi = 1$; hence the result.

One has now to check that it is stable for the $\rho_{S,K}$-majority rule under the p0, p1 and a0-governances. For that, given that Lemma 2 and 3 are still valid, we compute, as in Step 3 in the proof of Proposition 1, the relative volume, in Δ_S endowed with the density d_c, of the set defined by inequality (13) which is, denoting $u = \pi_{S_1 \setminus 1}$ and $v = \pi_{(S \setminus S_1) \cup T}$:

$$(cS-1)! \int_{u=0}^{\frac{n}{n+1}} \frac{u^{cn-1}}{(cn-1)!} \left(\int_{v=0}^{1-\frac{n+1}{n}u} \frac{v^{c(S-n-1)-1}}{[c(S-n-1)-1]!} \frac{(1-u-v)^{c-1}}{(c-1)!}\, dv \right) du\,;$$

which is equal, after standard integration by parts, to $1-\rho_{S,K}$.

As far as the a1-governance is concerned, one has to compute the volume of the set defined by inequality (13), endowed with the density d_c, and moreover weighted by π_{S_1} (the same result would hold for any other firm). This is

$$(cS-1)! \int_{u=0}^{\frac{n}{n+1}} \frac{u^{cn-1}}{(cn-1)!} \left(\int_{v=0}^{1-\frac{n+1}{n}u} \frac{v^{c(S-n-1)-1}}{[c(S-n-1)-1]!} \frac{(1-u-v)^{c-1}}{(c-1)!} (1-v)\, dv \right) du\,;$$

which (we cut this double integral into two pieces by distributing with respect to $1-v$) is equal to $(1-\rho_{S,K}) + \frac{c(S-n-1)}{cS}(1-\rho_{S,K}) = (1-\rho_{S,K})\frac{n+1}{S}$. On the other hand, the volume of Δ_S endowed with the density d_c and moreover weighted by π_{S_1} is $\frac{n+1}{S}$. Hence the result. $\square$

Proof of Corollary 2: This goes either by noticing that, for $n = 1$, for all c,

$$1-\rho_{S,K} = \left(\frac{1}{2}\right)^c \sum_{j=0}^{\infty} \binom{c-1+j}{j} \left(\frac{1}{2}\right)^j = \frac{1}{2}\,;$$

or by reproducing straightforwardly the proof of Proposition 2 given in Section 4. $\square$

Proof of Corollary 3: This goes by noticing that $\left(\frac{n}{n+1}\right)^{cn}\left(\sum_{j=0}^{\infty}\binom{nc-1+j}{j}\left(\frac{1}{n+1}\right)^{j}\right)$; decreases toward $\left(\sum_{i=0}^{\infty}\frac{c^i}{i!}\right)e^{-c}$, when n tends toward infinity. □

Proof of Proposition 5: It goes through three steps. The first step is the construction of a PE $(\hat{Y}; 1_K; \hat{\theta})$ based on the multiplan (Y^1); the second step proves that this PE is a $\tau_{S,K}$–MSPE; the third step proves that $\tau_{S,K}$ is the min-max majority.

Step 1: Suppose the K firms announce the production multiplan $\hat{Y}(\alpha) = [\hat{y}_k(\alpha)]_{k=1}^K$, for $\alpha \in R$:

$$\text{for } k \leq m;\ \hat{y}_k(\alpha) = (\alpha+1)\, y^1_k - \frac{\alpha}{K-m}\sum_{j=m+1}^{K} y^1_j\ ;\ \text{and for } k \geq m+1;\ \hat{y}_k = y^1_k\ : \tag{20}$$

Hence, as in the proof of Proposition 1, for $s \in S_k$, one has:

$$\text{if}\quad s \in T_k;\ k \geq m+1;\quad \text{then}\quad \sum_{j=1}^{K}\theta_j\,\hat{y}^s_j = \frac{1}{n}\left(\theta_k - \frac{\alpha}{K-m}\theta^S\right)$$

$$\text{if}\quad s \in S_k;\ k \leq m;\quad \text{then}\quad \sum_{j=1}^{K}\theta_j\,\hat{y}^s_j = \frac{\alpha+1}{n+1}\theta_k$$

Let $\pi(k) = \max_{s \in S_k\ (\text{or } T_k)} \pi^s$, the utility function (10), with $\pi^0(k) = \begin{cases}(n+1)\pi(k) & \text{if} \quad k \leq m\\ n\pi(k) & \text{if} \quad k \geq m+1\end{cases}$, can be rewritten:

$$\hat{U}_{\pi-1}[\theta(\pi)] = \min_k\left[\left(\frac{\alpha\theta_k}{\pi^0(k)}\right)_{k\leq m};\ \left(\frac{\theta_k - \frac{\alpha}{K-m}\theta^S}{\pi^0(k)}\right)_{k\geq m+1}\right]:$$

First order conditions at the maximum entails that all arguments of $\hat{U}_\pi$ are equal: there exists a real $c(\pi)$ such that,

$$k \leq m \qquad \hat{\theta}_k \quad = \quad \frac{1}{\alpha+1}\,\pi^0(k)\ c(\pi) \tag{21}$$

$$k \geq m+1 \quad \hat{\theta}_k \quad = \quad \left(\pi^0(k) + \frac{1}{K-m}\,\frac{\alpha}{\alpha+1}\sum_{j=1}^{m}\pi^0(j)\right)c(\pi) \tag{22}$$

We are looking for a value of α such that, at the equilibrium, $q_k = 1$, all k. As argued in the canonical case, since we consider homethetic preferences, there is no loss of generality in considering distributions of initial shares such that for all π $\sum_{k=1}^{K}\theta^0_k(\pi) = 1$ (together with the price normalization $\sum_{k=1}^{K} q_k = K$). Hence, if $q = 1_K$, one gets:

$$c(\pi) = \frac{1}{\sum_{k=1}^{K}\pi^0(k)}: \tag{23}$$

The question then is whether there exists a value of α such that for all k, $1 \leq k \leq K$,

$$\int_{\Delta_S} f(\pi)\,\hat{\theta}_k(\pi)\,d\pi = \int_{\Delta_S} f(\pi)\,\theta^0_k(\pi)\,d\pi:$$

For $k \leq m$, this last equation can be rewritten:

$$\frac{1}{\alpha+1}\int_{\Delta_S} f(\pi)\,\frac{\pi^0(k)}{\sum_{j=1}^{K}\pi^0(j)}\,d\pi = \int_{\Delta_S} f(\pi)\,\theta^0_k(\pi)\,d\pi = \frac{1}{K}; \tag{24}$$

this last equality being a direct consequence of the fact that $\int_{\Delta_S} f(\pi)\,\theta_k^0(\pi)\,d\pi = \Gamma_k^0$, independent of k, and that for all π $\sum_{k=1}^{K}\theta_k^0(\pi) = 1$. This defines obviously a value $\bar\alpha$ for the parameter α (which does not depend on k, because of the symmetry of the m first assets). And then one can easily check that (24) entails equilibrium on the markets for shares of firms k, $k \geq m+1$.

What remains to be checked is that $\bar\alpha \geq 0$ so that, at equilibrium, everybody is allocated positive shares of all firms. One obviously has, for all $k \leq m$:

$$\frac{\pi^0(k)}{\sum_{j=1}^{K}\pi^0(j)} \geq \frac{\pi(k)}{\sum_{j=1}^{K}\pi(j)};$$

therefore

$$\bar\alpha = K\int_{\Delta_S} f(\pi)\,\frac{\pi^0(1)}{\sum_{j=1}^{K}\pi^0(j)}\,d\pi - 1 \geq K\int_{\Delta_S} f(\pi)\,\frac{\pi(1)}{\sum_{j=1}^{K}\pi(j)}\,d\pi - 1$$

and since

$$1 = \sum_{k=1}^{K}\int_{\Delta_S} f(\pi)\,\frac{\pi(k)}{\sum_{j=1}^{K}\pi(j)}\,d\pi \underbrace{=}_{\text{by symmetry}} K\int_{\Delta_S} f(\pi)\,\frac{\pi(1)}{\sum_{j=1}^{K}\pi(j)}\,d\pi;$$

the result obtains. Fix $\alpha = \bar\alpha$ in the sequel.

Finally, one gets the following observation, with for all k, $\bar\mu_k = \frac{\pi^0(k)}{\sum_{j=1}^{K}\pi^0(j)}$.

Observation 3 One has:

$$\sum_{k=1}^{K}\hat\mu_k = \sum_{k=1}^{K}\bar\mu_k = 1; \tag{25}$$

and:

$$\sum_{k=1}^{K}\hat\mu_k\,\hat y_k = \sum_{k=1}^{K}\bar\mu_k\,\bar y_k. \tag{26}$$

Step 2: We have proved (for $\alpha = \bar\alpha$) that $(\hat Y; 1_K; \hat\mu)$ is a PE. We have to prove that it is $\tau_{S;K}$-majority stable. As for the canonical case, from the concavity of the agents preferences, for a fixed z_k, then $z_k(\lambda) = \lambda\hat y_k + (1-\lambda)z_k$ will get a larger or equal fraction than z_k of shareholders and shares against $\hat y_k$. Therefore, in looking for $P_{\hat Y;\hat\mu}(\hat y_k)$ or $A_{\hat Y;\hat\mu}(\hat y_k)$, we can focus on alternative production plan z_k arbitrarily close to $\hat y_k$. Given Observation 3, it is equivalent to take any z_k arbitrarily close to $\bar y_k$

Take $z_k = \bar y_k + \varepsilon_k$ with $\varepsilon_k = (\varepsilon_k^s)_{s=1}^{S}$, $\sum_{s=1}^{S}\varepsilon_k^s = 0$. Suppose $\varepsilon_k^1 < 0$: z_k reduces the payoff of $\bar y_k$ in the first state of the world. Necessarily, for all agents such that $\pi(1) = \pi^1$, the utility level decreases. Indeed, at the PE, the utility level of such an agent is $\frac{\bar\mu_1}{(n+1)\pi^1}$, and $U_{\pi^{-1}}[\bar x + \bar\mu_k\varepsilon_k]$ is by definition of the min inferior to $\frac{\bar\mu_1}{(n+1)\pi^1} + \frac{\bar\mu_k}{\pi^1}\varepsilon_k^1$. Take a state of nature s such that $s \in S_j$ (or T_j); then all shareholders π such that $\pi(j) = \pi^s$ will strictly oppose to any change ε_k such that $\varepsilon_k^s < 0$.

Therefore, in order to find a best challenger against the status quo $\bar y_k$ within firm k, one has to decrease the payoffs in the smallest number of states of the world, hence only one: Fix $\bar s$ such that $\bar s \in S_{\bar j}$ (w.l.o.g.) and consider $\varepsilon_k^s > 0$ for all $s \neq \bar s$, and $\varepsilon_k^{\bar s} < 0$. The change ε_k imapirs the welfare of shareholders π such that $\pi(\bar j) = \pi^{\bar s}$, but on the other hand, for ε_k sufficiently small, it improves the welfare of all other agents. Indeed, consider an agent π such that $\pi(\bar j) \neq \pi^{\bar s}$. Then in states $s \neq \bar s$, its consumption increases by $\frac{\hat{\mu}_k}{¼_{\hat{s}}}\, ²^{s}_{k}$; and in state $\hat{s}$ its consumption decreases by $\frac{\hat{\mu}_k}{¼_{\hat{s}}}\, ²^{\hat{s}}_{k}$. But since $¼(\hat{j}) > ¼_{\hat{s}}$, for $²_k$ small enough, one has:

$$\frac{\hat{\mu}_k}{¼_{\hat{s}}}\,\frac{1}{n+1} + \frac{\hat{\mu}_k}{¼_{\hat{s}}}\, ²^{\hat{s}}_{k} > \frac{\hat{\mu}_k}{¼(j)}\,\frac{1}{n+1} + \frac{\hat{\mu}_k}{¼(j)}\, ²^{\hat{s}}_{k}\;;$$

where $\hat{s}$ is such that $¼(\hat{j}) = ¼_{\hat{s}}$. Hence, even after the change $²_k$, the minimum consumption still does not occur for $\hat{s}$, but for another state s in which the consumption increases.

Therefore the most challenging way to threaten the status quo is to decrease the payo® of $\hat{y}_k$ for only one state of the world and improving its payo® in all others. In this case the shareholders opposing the change are such that $¼(\hat{j}) = ¼_{\hat{s}}$. Given the symmetry of the density f, their proportion is obviously $\frac{n}{n+1}$ if $\hat{s} \cdot m(n+1)$ (i.e., $\hat{s}$ 2 $S_{\hat{j}}$ for some $\hat{j} \cdot m$), and $\frac{n¡1}{n}$¥ if $\hat{s}_{¸}$ $m(n+1)+1$ (i.e., $\hat{s}$ 2 $T_{\hat{j}}$ for some $\hat{j}$ ¸ $m+1$). Hence $P_{\hat{Y};\mu^0}(\hat{y}_k) = P_{\hat{Y};\hat{\mu}}(\hat{y}_k) = \frac{n}{n+1}$ with $n =$ ¥$\frac{S¡1}{K}$¦. Finally, given the symmetry, at the considered symmetric PE, of the shares, one immediately gets $A_{\hat{Y};\mu^0}(\hat{y}_k) = A_{\hat{Y};\hat{\mu}}(\hat{y}_k) = \frac{n}{n+1}$.

Step 3: We have proven that there exists a $¿_{S;K}$-MSPE for all governances. Finally we have to prove that $¿_{S;K}$ is the min-max.This easy proof is left to the reader: take a non-symmetric Y and `attack' it the usual way from the central part of $¢_S$. 2