Waiting for my Neighbors

Sidartha Gordon, Emeric Henry, Pauli Murto

Discussion Paper

WAITING FOR MY NEIGHBORS

Sidartha Gordon, Emeric Henry, and Pauli Murto

SCIENCES PO ECONOMICS DISCUSSION PAPER

No. 2018-10

Waiting for my neighbors

Sidartha Gordon*, Emeric Henry† and Pauli Murto‡

August 31, 2018

Abstract

We introduce a neighborhood structure in a waiting game, where the payoff of stopping increases each time a neighbor stops. We show that the dynamic evolution of the network depends on initial parameters and can take the form of either a shrinking network, where players at the edges stop first, or a fragmenting network where interior players stop first. We find that, in addition to the coordination inefficiency standard in waiting games, the neighborhood structure gives rise to two other inefficiencies, the first linked to the order of exit and the second to the final distribution of remaining nodes.

JEL Classification: D85, C73, D83

Keywords: waiting games, networks, inefficiencies

1 Introduction

There is growing evidence that the decision to adopt a new technology is affected by the decisions of neighbors, i.e those close either geographically or in terms of social or technological distance (Foster and Rosenzweig 1995, Conley and Udry 2010, Bandiera and Rasul 2006, Atkin et al. 2017). One explanation is that adoption creates spillovers for neighbors that decrease their own adoption costs. These spillovers can be informational or technological. For instance, the initial adopter trains employees or suppliers with this new technology and the mobility of workers or the sharing of suppliers spreads the expertise to connected firms.

Such environments create incentives for players to wait for their neighbors to adopt. In this paper we introduce a class of problems, *waiting games on networks*, that encompasses the adoption problem described above. There are other applications besides

*LEDa, University Paris Dauphine and PSL Research University, sidartha.gordon@dauphine.fr.
†Sciences Po, emeric.henry@sciencespo.fr.
‡Aalto University School of Business, pauli.murto@aalto.fi.

technology adoption for such models, such as industry shakeouts where only one firm can survive in a neighborhood and firms wait in the hope that neighbors exit first. Such war of attrition games have been extensively studied, but the neighborhood structure has to this point been ignored. We show that it leads to two new sources of inefficiencies in addition to the usual timing inefficiency arising in classical war of attrition games. The first, linked to the order of stopping of players and the second to the final distribution of nodes.

To highlight these new sources of inefficiencies, we use the simplest neighborhood structure which is the line, where each player has either one or two neighbors. We discuss larger networks in the conclusion. Specifically, players are organized on line segments of random length and play an infinite horizon timing game. Each player has to decide when to take an action, we call "stop". The benefit of the action for an individual at date t depends on the neighbors' past actions. Whenever a player stops, she increases the payoff of stopping of all her neighbors. This creates incentives for all players to wait in the hope that their neighbor(s) stop before them, i.e gives rise to the structure of a waiting game.

Each link between two consecutive players is i.i.d drawn at date 0. The probability distribution of the network structure is common knowledge, but players do not observe the realization of the network structure but only their direct neighbors (as in Jackson and Yariv 2005, 2007 or Galeotti et. al. 2010).[1] This implies that a player does not know which of two possible types a given neighbor is: either the neighbor has one neighbor (i.e. she is at the end of the line), or two neighbors (i.e. she is inside the line). We restrict ourselves to symmetric strategies and show that at any point in the game, players share the same belief about the type of an arbitrary neighbor. The endogenous evolution of these beliefs is the key aspect of our analysis as we explain below.

Generically, in a symmetric equilibrium of our game, at any given date, only one type of player mixes between stopping and waiting, while players of the other type strictly prefer to wait. Two very different dynamic evolutions of the network can emerge based on parameters of the model, and in particular on the payoffs of the different types. First, what we call shrinking networks, where players of type 1 (extremities of the line) initially have more incentives to stop and hence the network shrinks over time. Second, fragmenting networks where players of type 2 (inside the line) initially have more incentives to stop, which leads to a fragmentation of the network in smaller networks over time.

[1] For our applications this assumption captures the idea that players are not aware of the full structure of the network.

Consider first shrinking networks. Initially, players of type 1 are mixing. As time passes and the unique neighbor of a player has not stopped, the belief about the type of the neighbor evolves. Two countervailing forces affect this belief. First, there is the classic updating of beliefs: since players of type 1 are more likely to stop, as time passes, the player becomes more confident that the neighbor is of type 2. However, there is a second countervailing effect, purely linked to the dynamic evolution of the network structure. Even if the neighbor started off as a type 2, her other neighbor might have stopped in the meantime, making it possible that she has turned into a type 1. We show that these two effects perfectly balance each other in the line network, so that the beliefs that the player is of type 1 stays constant throughout the game and only players at the extremity of the line mix and do so at a constant rate, as if they were playing a classic war of attrition with a single player of a given type.

When players of type 2 initially have more incentives to stop, we have the case of fragmenting networks, and both effects affecting beliefs mentioned above go in the same direction. As time passes and a neighbor has not stopped, players become more confident that she is of type 1. In addition, even if she started as a type 2, her own neighbor might have stopped, changing her into a type 1. Thus, as time passes, the belief that the neighbor is of type 2 decreases. Over time the network fragments into smaller networks. At some date, all players of type 2 will have stopped and only isolated pairs will remain. These pairs will then play a classical war of attrition.

Waiting games give rise to a timing inefficiency: players inefficiently delay stopping to potentially benefit from the action of others. A key message of the paper is to show that the network structure gives rise to additional sources of inefficiency, that we call *order inefficiency* and *spatial inefficiency.* The first is linked to the order in which players stop. Players, when they decide whether to stop, do not take into account the positive externality they provide to their neighbors. This implies that, in equilibrium, players inside the line, i.e with more neighbors, have insufficient incentives to stop. There is thus a region of parameters where the equilibrium results in a shrinking network whereas the first best would be a fragmenting network. We also show that even in the case of the fragmenting network, the order of exit of players inside the line is important for total welfare. We distinguish *regular fragmenting* (where every other player inside the line exits) from *random fragmenting* where any player inside the line is equally likely to stop.

Random fragmenting, which is the stopping process in equilibrium for fragmenting networks, leads to the second additional inefficiency compared to waiting games without networks, a *spatial inefficiency* linked to the distribution of nodes at the end of the game. When we compute the total fraction of nodes that remain at the end of the game we find that it is strictly less that 1/2 (that would occur under regular fragmenting) and that the probability of having a gap of size 2 between nodes is relatively large. We argue that this final distribution of nodes can be socially costly. For instance, in the case of industry shakeouts it can be socially optimal to have equally spaced firms with minimum gaps between them if for instance customers are uniformly distributed and pay transport costs.

In the final part of the paper, we consider in more detail the application to adoption of technologies by firms organized in a network and consider subsidies for adoption, common in practice, that can mitigate the timing inefficiency inherent in any war of attrition. The network structure adds to the effectiveness of such policies. Since every adoption decision entails a positive externality on *all* subsequent adopters, the positive welfare effect of the subsidy propagates through the network. We compare different types of subsidy programs financed by distortionnary taxes. A permanent subsidy program targeted uniformly to all adopters can, if the social cost of taxes is not too high, increase welfare by accelerating adoption and thus partially solves the coordination problem. However, we show that there are more efficient ways to accelerate adoption, by using time varying policies. We examine two such policies: a randomly expiring subsidy that captures the fact that any subsidy program is subject to reform (either because of political turnover or economic shock) and a smoothly declining subsidy. Both dominate permanent subsidy as they harness the idea that if the subsidy payment is lower tomorrow than it is today, it provides incentives to immediately adopt. Finally, we consider a "neighbor reward policy", where a subsidy is paid to a player at the moment when one of her neighbors adopts after she has adopted herself. We show that such a policy, if feasible, can target the subsidies more efficiently and hence increase adoption incentives at a lower social cost than the uniform subsidy programs.

As Jackson and Zenou (2014) point out, the literature on strategic dynamic games on networks is still limited, and in particular there are no infinite horizon strategic timing games on networks. Most interest has in fact focused on repeated games (Raub and Weesie 1990, Ali and Miller 2013 among others). The core of the mechanism is that punishment of deviations by one neighbor will also impact the payoff of the other neighbors and contagion of bad behavior can thus occur.

Leduc et. al. (2017) consider a related problem, but focus on information diffusion in a two period model. Players need to take an action whose payoff depends on a binary state. If a player takes the action in period 1, all her neighbors learn the state, and the player obtains a referral payoff. The authors solve for the mean-field approximation of the game, as in much of the literature on network diffusion (such as in Jackson and Yariv 2007), and show that agents with low degree have incentives to take the action early while those with higher degrees free ride. We are interested in the fully dynamic game where payoffs depend on the number of neighbors and we solve directly for the perfect Bayesian equilibrium. This allows us to introduce learning about the network structure as the game evolves. We explicitly uncover two new sources of inefficiencies, due to the final distribution of nodes and to the order of exit.

The literature on war of attrition games has been applied to many cases, including the original work on biological competition (Maynard Smith 1974), labor strikes (Kennan and Wilson 1989), industrial organization (Fudenberg and Tirole 1986). Bulow and Klemperer (1999) consider a generalization of the classic model to the case of $n+k$ players competing for n prizes. None of the papers consider the influence of the neighborhood structure, which is the focus of the current study.

2 Model

There is a countable set of players labeled by $i \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$. Each pair of consecutive players i and $i+1$ are initially connected to each other with probability $\chi \in (0,1)$, independently of all other consecutive pairs. Hence, the players are organized in a countable set of finite segments, where the length of each segment follows a geometric distribution. We say that two players are neighbors if they are linked to each other. Each player can be of type $k \in \{0, 1, 2\}$, where k denotes the number of her neighbors. We can express the fraction of the players that are initially of type k in terms of χ as:

$$\begin{aligned} q_0 &= (1-\chi)^2, \\ q_1 &= 2\chi(1-\chi), \\ q_2 &= \chi^2. \end{aligned} \tag{1}$$

Players only have to decide when to take an action that we call “stop”. Time is continuous and the benefit for an individual stopping at date t depends on how many neighbors she has at that date. If a player stops at time t, and has k neighbors at that date, her realized payoff is

$$\Pi(k,t) = e^{-rt} B_k,$$

where r is the rate of discounting and B_k is the time invariant benefit of stopping for a player with k neighbors. We are interested in a class of games where B_k is a decreasing sequence ($B_2 < B_1 < B_0$). We present foundations for this payoff structure in Section 5.

The structure of the neighborhood evolves dynamically. As soon as a player stops, she exits the game. We represent this as a deletion of all her links. Consider a player with initially k neighbors, so that initially her payoff if she decided to stop would be B_k. If one of her neighbor stops, she is left with $k-1$ neighbors, and her payoff of stopping increases to B_{k-1}. This creates incentives for all players to wait for others to stop.

2.1 Strategies and information

Each player observes her own neighbors, shares a common prior on the network structure but has incomplete information about the realized structure (as in Galeotti et. al. 2010). Since the players only observe the behavior of their neighbors, a private history at t for a player consists of stopping dates of her neighbors up to time t.

It is immediately clear that for a player of type $k = 0$ who has no neighbors, it is strictly dominant to stop immediately. For notational simplicity we ignore type $k = 0$ in the definition of strategies.

Consider the strategy for player i of type $k = 1$ who initially has one neighbor. A pure strategy for such a player is simply a stopping time i.e. $T_1^i \in [0, \infty)$. This stopping time is conditional on her neighbor still remaining in the game: if the only neighbor stops at time $t < T_1^i$, i becomes type $k = 0$ and stops also at time t.

A player of type $k = 2$ who initially has two neighbors has two components in her strategy. First, she must choose when to stop conditional on none of her two neighbors having stopped, i.e. choose $T_2^i \in [0, \infty)$. If one of the two neighbors stops before T_2^i, she becomes type $k = 1$ and must choose when to stop conditioning on the time at which she became type $k = 1$. Therefore, the other component of a pure strategy is a mapping $T_{21}^i(\tau) : [0, \infty) \to [0, \infty)$ that defines the time to stop when one of the two neighbors stops at time τ. This mapping must satisfy $T_{21}^i(\tau) \geq \tau$ for all $\tau \geq 0$. The entire pure strategy for player i can hence be written as

$$T^i = \left(T_1^i, \left(T_2^i, T_{21}^{i^i}(\cdot)\right)\right),$$

and a corresponding behavioral strategy is

$$\sigma^i = \left(\sigma_1^i, \left(\sigma_2^i, \left\{\sigma_{21}^i(\tau)\right\}_{\tau \geq 0}\right)\right),$$

where σ_1^i and σ_2^i are probability distributions over $[0, \infty)$ and $\sigma_{21}^i(\tau)$ is a probability distribution over $[\tau, \infty)$.

Given a profile σ of behavioral strategies, the outcome of the game consists of realized payoffs and stopping dates and is defined by the following algorithm. Take any finite segment of players, and proceed through the following steps:

Step 1: For each player i in the line segment, draw the planned stopping date t^i from the appropriate component of the behavioral strategy (from component σ_1^i if i has one neighbor, or from component σ_2^i if i has two neighbors). Denote the minimum of these stopping times across the players by t, and let each player who chose that time stop at t and get payoff $e^{-rt}B_k$. If, as a result, some of the remaining players becomes type $k = 0$, let those players also stop at t and get payoff $e^{-rt}B_0$. Proceed to step 2.

Step $n \geq 2$: Amongst those players that remain after step $n-1$, take all such players i that were originally type $k = 2$ but became type $k = 1$ and draw for them a new planned stopping time t^i from $\sigma_{21}^i(t)$. For all the other remaining players keep their planned stopping time unchanged. Then, repeat the same procedure as in step 1, i.e. take again the minimum of the planned stopping times amongst the remaining players, remove those players who chose that stopping time and give them appropriate payoffs, remove all players that became type $k = 0$. If there are players left, go to step $n + 1$.

Since the number of players is finite, all the players will have stopped after a finite number of steps at well defined dates and obtained their payoffs.

2.2 Beliefs

The probability distribution of the network structure is common knowledge to the players, but each player only observes her own neighbors. Each player therefore forms an initial belief on the type of her neighbor(s), either $k = 1$ or $k = 2$. Since a link exists between any two consecutive players independently with probability χ, the initial belief of an arbitrary player about the type of an arbitrary neighbor is:

$$\begin{aligned} p_1 &= 1 - \chi, \\ p_2 &= 1 - p_1 = \chi. \end{aligned}$$

Since there is one-to-one correspondence between parameter χ and initial beliefs p_1 and p_2, we eliminate χ from the rest of the analysis and track only the players' beliefs about their neighbors' types. Using (1), we can also express the initial beliefs as functions of the initial fraction of different types of players as:

$$p_1 = \frac{q_1}{q_1 + 2q_2},$$
$$p_2 = \frac{2q_2}{q_1 + 2q_2}.$$

There are two key properties of this initial belief structure, $P1$ and $P2$, that are important for our analysis.

Property $P1$ **(Anonymity)**: A player $i \in \mathbb{Z}$ has the same belief on the type of each of her neighbors (if she has any), and this belief is the same for all i, regardless of her own type.

Property $P2$ **(Independence):** If a player has two neighbors, she believes that their types are independently distributed.

As time passes, the players update their beliefs on their neighbors' types based on the equilibrium strategies. We establish in the following Lemma that the two properties $P1$ and $P2$ continue to hold for any date $t > 0$ as long as players use symmetric strategies.

Lemma 1 *With symmetric strategy profiles,* $P1$ *and* $P2$ *remain satisfied at all dates* $t \geq 0$.

This result allows us to summarize the belief structure at time t by a vector $p(t) = (p_1(t), p_2(t))$, where $p_2(t) = 1 - p_1(t)$.[2]

2.3 Equilibrium

Our solution concept is perfect Bayesian equilibrium. Throughout the paper we treat the players anonymously, and therefore concentrate on symmetric strategy profiles. A strategy profile σ is a symmetric perfect Bayesian equilibrium if it is symmetric and (i) the belief $p(t)$ about a neighbor's type is derived from σ via Bayes' rule for all private histories and (ii) the strategy σ is optimal for any private history given the belief $p(t)$ and given that all the other players use the strategy σ.

Although an arbitrary strategy profile is a complex object, a symmetric equilibrium profile can be summarized by the distribution of stopping dates that it induces for any given neighbor of i conditional on i never stopping. By symmetry and Lemma 1, this

[2] Note that even if the belief structure can be expressed as this simple object, it implies a belief on the shape of the entire network. In particular, each player believes that the length of the half-line starting from a neighbor is geometrically distributed with parameter $p_2(t)$.

distribution is the same for all neighbors of i and for all i. Let this distribution be denoted by F. In the following Lemma we summarize a key property of F that holds in any symmetric equilibrium:

Lemma 2 *In a symmetric equilibrium, the stopping date of an arbitrary neighbor has an atomless probability distribution F with full support on* $[0, \infty)$.

Lemma 2 allows us to define for all $t \geq 0$ the hazard rate of stopping of an arbitrary neighbor as

$$\gamma(t) = \frac{F'(t)}{1 - F(t)}.$$

In general, the same hazard rate γ can sometimes admit multiple type breakdowns. In order to reduce this multiplicity, we restrict attention to **Markovian** equilibria such that players who are type 1 at date t and who started off as types 2 play the same continuation strategy from date t regardless of the date at which their neighbor stopped, and also the same continuation strategy from date t as a player who started off as a type 1 from date 0.

In a Markovian equilibrium, the hazard rate γ can be uniquely broken down by type. For each $k \in \{1, 2\}$ such that $p_k(t) > 0$, there exists a function $\lambda_k(t)$ that gives the hazard rate of stopping of a player of type k at date t.[3] Moreover,

$$\gamma(t) = p_1(t)\lambda_1(t) + p_2(t)\lambda_2(t).$$

In what follows, we will characterize Markovian symmetric equilibria of the model by directly analyzing the properties of the stopping hazard rates λ_k for each type $k = 1, 2$. We show in Appendix C that our main results extend to non Markovian strategies.

3 Waiting for my neighbors: equilibrium characterization

In our model, the heterogeneity between players is due to their position in the line, specifically the number of neighbors that they have. To understand the specific role of the neighborhood structure, we first study a waiting game with heterogenous types, where the source of heterogeneity is not linked to a particular neighborhood structure.

[3]For types k such that $p_k(t) = 0$, the function $\lambda_k(t)$ is indeterminate.

3.1 Benchmark with no neighborhood structure

We consider a game between two players who can have one of two possible types: type 1 gets benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 gets benefit B_2 if first and B_1 if second ($B_0 > B_1 > B_2$). Both players know their type and share a common prior that the other player is of type $j \in \{1, 2\}$ with probability p_j. Consistent with our model with neighbors, the belief about the other player's type is independent of one's own type. We derive the symmetric equilibrium of this game. The shape of the equilibrium depends on the comparison between μ_1 and μ_2 where $\mu_j = \frac{rB_j}{B_{j-1}-B_j}$.

Proposition 1 *If $\mu_j > \mu_k$ (either $j = 1$ and $k = 2$ or the reverse), there exits a date t_j^b such that in all symmetric equilibria:*

- *For $t < t_j^b$ only players of type j mix between the actions stop and wait. Both players expect the other to stop at a rate μ_j.*
- *The posterior belief that the other player is of type j, $p_j(t)$, is decreasing and such that $p_j(t_j^b) = 0$.*
- *For $t \geq t_j^b$ players of type k mix at constant rate μ_k.*

As shown in Proposition 1, in a symmetric equilibrium, only one single type mixes at any point in time. Indeed, when a player of a given type $l \in \{1, 2\}$ is mixing, she needs to be indifferent between the cost of waiting, equal to rB_l and the expected gain if the other player stops, equal to $(B_{l-1} - B_l)$ that accrues with probability μ, where μ is the rate of stopping of the other player. Since types are not correlated, μ is independent of the own type and generically only one type can satisfy the indifference condition

$$\mu\,(B_l - B_{l-1}) = rB_l. \tag{2}$$

Proposition 1 then characterizes the timing of actions. Consider the case where $\mu_1 \equiv \frac{rB_1}{B_0-B_1} > \mu_2 \equiv \frac{rB_2}{B_1-B_2}$. Players of type 1 have more incentives to stop and initially are the only types to mix. The equilibrium mixing rate, as can be seen in equation (2), has to be such that all players share the belief that the other player will stop at rate $\mu = \mu_1$. Note that μ_1 is both a function of the belief that the other player is of type 1 and of the mixing rate λ_1 of players of type 1. We have specifically $\mu_1 = p_1(t)\lambda_1(t)$. As time passes and the other player has not stopped, the posterior $p_1(t)$ that she is of type 1 decreases. At some date t_1^b all types 1 will have stopped. If the two players are still active, they are then certain that the other is of type 2. Players of type 2 then start mixing at a constant rate μ_2 as in a classical war of attrition.

3.2 Waiting for my neighbors

We now explicitly introduce the neighborhood structure and the heterogeneity between players is then due to the position in the line. Types differ in the number of neighbors they have and thus in terms of payoffs when stopping. The payoffs when stopping are the same as for the benchmark studied above: type 1 who makes benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 who gets benefit B_2 if first and B_1 if second.

There are two key differences with the benchmark model. First, for types 2, the fact of having two neighbors doubles the chances of at least one of them stopping and thus affects the strategic choices. Second, and more important, the types evolve dynamically: if the neighbor of a given player is initially a type 2 but her other neighbor stops, she becomes a type 1. This change in the type of the neighbor is not observed by the player, but the possibility of such a dynamic evolution affects the beliefs about the neighbor's type.

It is important to distinguish two cases depending on the respective values of

$$\overline{\gamma}_1 := \frac{rB_1}{B_0 - B_1} \quad \text{and} \quad \overline{\gamma}_2 := \frac{rB_2}{2\,(B_1 - B_2)}.$$

We show that the case $\overline{\gamma}_1 > \overline{\gamma}_2$ is one where the players of type 1 mix first. This gives rise to what we call "shrinking networks" since only the players at the extremities of a line mix and over time the line gets shorter. On the contrary, in the case $\overline{\gamma}_2 > \overline{\gamma}_1$, players of type 2 have initially more incentives to mix. This gives rise to what we call "fragmenting networks". The initial line is cut at some date into two smaller segments and this process repeats itself over time.

Recall that in the benchmark model of section 3.1, two cases were distinguished based on the respective value of μ_1 and μ_2, which determined which type was mixing first (here we have $\overline{\gamma}_1 = \mu_1$ but $\overline{\gamma}_2$ is different from μ_2 since it integrates the fact that a type 2 has two neighbors in our current setup). However, both cases are qualitatively equivalent in the benchmark while in the case with a network structure, the two cases are radically different, due to the dynamic evolution of the network structure.

3.3 Shrinking networks

We start by considering the case $\overline{\gamma}_1 > \overline{\gamma}_2$. As in the benchmark model, only one type of player can be mixing at any point in time. In this case, players of type 1 have more incentives to mix and hence stop first.

However, the key difference with the benchmark case is that, as players of type 1 are mixing, two forces affect beliefs, as reflected in the following dynamic equation:

$$\dot{p}_1(t) = \underbrace{-\lambda_1(t)\, p_1(t)\,(1-p_1(t))}_{\text{updating beliefs about initial type}} + \underbrace{\gamma(t)\, p_2(t)}_{\text{probability that type 2 becomes 1}} \quad (3)$$

where, as defined in Section 2, $\lambda_1(t)$ is the hazard rate of stopping of a neighbor of type 1 and $\gamma(t)$ is the expected hazard rate of stopping of an arbitrary neighbor (so that $\gamma(t) = p_1(t)\,\lambda_1(t)$).

The evolution of beliefs described in (3) reflects the balance between two effects. First, players update their beliefs about their neighbor's types based on the fact they do not see her stopping. Second, the types of neighbors may evolve dynamically since even if the neighbor initially had two neighbors (probability $p_2(t)$), her other neighbor might have stopped in the time interval (probability $\gamma(t)$), thus changing her type into a type 1. The two effects on beliefs go in opposite direction. The first effect makes the player less confident that the neighbor started off as a type 1 but the second makes it more likely that she became one over time. The following result examines the balance between these effects in equilibrium:

Proposition 2 *If $\overline{\gamma}_1 > \overline{\gamma}_2$, a Markovian symmetric equilibrium has the following properties:*

1. *The belief that a random neighbor is of type 1 remains constant, equal to $p_1(0)$ throughout the game and type 1 players mix at constant rate $\lambda_1 = \frac{\overline{\gamma}_1}{p_1(0)}$.*
2. *The expected time before an average member of the network stops is given by $E[T] = (q_1 + q_2)\frac{1}{2\overline{\gamma}_1}$ and is increasing in B_0, decreasing in B_1 and independent of B_2.*

Proposition 2 shows that in the Markovian symmetric equilibrium[4] the two effects on beliefs perfectly balance each other in the case of the line.[5] As a consequence, the

[4]These properties are also satisfied in non Markovian symmetric equilibria, one example of which we provide in the Appendix.

[5]The fact that the two effects perfectly balance each other is specific to the line and is no longer true for networks where agents may have more than two neighbors.

belief that the neighbor is a type 1 stays constant throughout the game and players play an infinite war of attrition as if they were facing a single player mixing at rate $\overline{\gamma}_1$. Only the players of type 1, by definition positioned at the extremities of the line, mix at any point in time. Overall, the line shrinks in size over time, hence the terminology. Furthermore, as reflected in result 2, since only type 1 players stop during the entire game, decreasing their incentives to stop by increasing B_0 or decreasing B_1 delays stopping. Similarly, since type 1 incentives are independent of B_2, the equilibrium stopping rate is independent of B_2.[6]

3.4 Fragmenting networks

We now consider the case $\overline{\gamma}_2 > \overline{\gamma}_1$. We show that in this case, types 2 initially have the highest incentives to stop. As in the previous case the evolution of beliefs about the neighbor's type is the result of two effects: updating based on the fact that the neighbor did not stop and dynamic evolution of beliefs. However in this case both effects go in the same direction and as time passes it becomes increasingly likely that the neighbor is of type 1. Formally, the evolution of the beliefs is given by:

$$\dot{p}_2(t) = \underbrace{-\lambda_2(t)\, p_2(t)\,(1 - p_2(t))}_{\text{updating beliefs about initial type}} - \underbrace{\gamma(t)\, p_2(t)}_{\text{probability that type 2 becomes 1}} \tag{4}$$

where $\lambda_2(t)$ is the hazard rate of stopping of a neighbor of type 2 and $\gamma(t)$ is the expected hazard rate of stopping of an arbitrary neighbor (so that $\gamma(t) = p_2(t)\,\lambda_2(t)$).

As in the benchmark case of section 3.1, $p_2(t)$ is decreasing and at some date t_2 players are sure that their neighbor is not of type 2, i.e $p_2(t_2) = 0$. At that date, types 1 mix exactly as in the benchmark case.

The rate of stopping by types 2 does not however follow the same dynamics as in the benchmark case. If a type 2 player decides to stop, she gets B_2 as in the benchmark case. When she waits, it is in the hope that one of her two neighbors stops in the meantime, at which point she will become a type 1 with value $V_1(t)$ that varies over time (while it was constant in the benchmark). Thus the stopping rate of a random neighbor is given by:

$$\gamma(t) = \frac{rB_2}{2\,(V_1(t) - B_2)},$$

[6] In the extreme case where $B_0 = B_1 > B_2$, players stop one by one at date 0, and there is no delay. Indeed, a type 1 player has nothing to wait for in that case, and it is a dominant strategy for her to stop at date 0, as soon as she turns into a type 1.

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma(t) B_0 dt + (1 - \gamma(t) dt)(1 - rdt)\left(V_1(t) + \dot{V}_1(t) dt\right)$$

Indeed, the payoff of a player of type $k = 1$ at a date t where only types $k = 2$ are mixing is composed of the expected payoff in the period dt, which is B_0 if the neighbor stops (probability $\gamma(t)$), plus the continuation value. As long as players types $k = 1$ strictly prefer to wait, we have $V_1(t) > B_1$, but $V_1(t)$ is strictly decreasing in time.

Proposition 3 *If $\overline{\gamma}_2 > \overline{\gamma}_1$, there exits a date t_2 such that a (Markovian) symmetric equilibrium satisfies:*

- *For $t < t_2$ only types 2 are mixing and the expected rate of stopping of a random neighbor is $\gamma(t) = \frac{rB_2}{2(V_1(t) - B_2)}$, where $V_1(t)$ is the value function of type 1. We have $B_0 > V_1(t) > B_1$ and $\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0$.*
- *At time $t = t_2$, $V_1(t_2) = B_1$ and $p_2(t_2) = 0$.*
- *For $t > t_2$ players of type $k = 1$ mix at a constant hazard rate $\overline{\gamma}_1$.*

Furthermore, if $p_2(0) < \frac{1}{2}$, types 2 are active for a longer period of time than in the benchmark case ($t_2 > t_2^b$).

Compared to the benchmark model, there are two main forces that affect the time t where the players are sure the other player is not of type 2 (i.e t_2 in the case under consideration and t_2^b in the benchmark model). First, types 2 mix at a lower rate for two reasons: they have two neighbors, so the chance of at least one stopping is higher than in the benchmark model. Furthermore, the value obtained if one neighbor stops, V_1, is higher than in the benchmark, B_1. Both these effects imply that there are more incentives to wait and the stopping rate will be lower. At the same time, as time passes, some neighbors of type 2 become type 1 thus decreasing the incentives to wait. If the proportion of types 2 is initially small as indicated in the last result of Proposition 3, the first effect dominates.

The dynamic evolution is very different than in the case of the shrinking network. Only types 2, situated at the heart of the network as opposed to its extremities, initially mix. At some point one of them randomly stops. The initial network is then fragmented in two smaller networks and the same process repeats itself. We explore in section 4.2

the consequences of this fragmentation process in terms of spatial distribution of players at the end of the game.[7]

4 Inefficiencies in waiting games on networks

Standard waiting games lead to inefficient delays due to coordination failures. We explore this inefficiency in the case of technology adoption among neighbors and possible solutions using subsidies in section 5. However, the network structure gives rise to two additional inefficiencies. An order inefficiency, studied in section 4.1, that relates to the order in which players stop and a spatial inefficiency, studied in section 4.2, linked to the final spatial distribution of nodes.

4.1 Order inefficiency

The previous sections established that, in equilibrium, the network behaves as a shrinking network rather than a fragmenting network if and only if

$$\begin{aligned} \overline{\gamma}_1 \geq \overline{\gamma}_2 \quad &\Leftrightarrow \quad \frac{B_1}{B_0 - B_1} \geq \frac{B_2}{2\,(B_1 - B_2)} \\ &\Leftrightarrow \quad 2B_1 \geq B_2 + \frac{B_2}{B_1} B_0 \ . \end{aligned} \tag{5}$$

We show in this section that this order of stopping can be socially inefficient. When a player decides to stop, she provides a positive externality to her neighbors as their own payoffs from stopping increase. Because players do not internalize this externality, players with more neighbors might have insufficient incentives to stop. This implies that the players might in equilibrium behave as in a shrinking network while the first best would be a fragmenting network, where the players with more neighbors stop first.[8]

Moreover, the order of exit in the fragmenting case also has an impact on welfare. There are different possible stopping orders that would result in different degrees of fragmentation of the network. Consider two different modes of stopping, which we call *regular fragmenting* where exactly every second player exits before their neighbors (for instance even numbered players stop first) and *random fragmenting*, where any interior player of type 2 can be the first to stop. Regular fragmenting is the stopping order that

[7] It is worth noting also that unlike the extreme case we considered for shrinking networks, in the extreme case $B_2 = B_1 < B_0$, type 2 players do not exit at date 0 in equilibrium. The equilibrium remains qualitatively similar to the generic case studied in Proposition 3. Players 2 remain active until a date $t_2 > 0$. They wait for both of their neighbors to exit.

[8] Note that the timing inefficiency is also the result of ignored externalities.

maximizes the number of players that get payoff B_2 and minimizes the average distance between players. Obtaining regular fragmenting requires information on the exact network details available to the planner but not to the players. Random fragmenting is the equilibrium stopping order in the case of fragmenting networks. We will analyze in more detail the statistical properties of the final distribution of nodes in the case of random fragmenting in Section 4.2.

We contrast in Proposition 4 the socially optimal stopping order to the equilibrium order, where by socially optimal we mean the order that maximizes the sum of the players' payoffs ignoring any costs of delay. We show that whenever $2B_1 \geq B_2 + B_0$, the social planner wants to minimize the degree of fragmentation, i.e. minimize the number of players that get payoff B_2. Then the shrinking network is the best while a regular fragmenting network is the worst. Conversely, when the condition is violated, the planner wants to maximize the degree of fragmentation. Then the regular fragmenting network gives the highest welfare while the shrinking network is the worst. In both cases the random fragmenting network gives an intermediate level of total welfare. We summarize this discussion in the proposition below.

Proposition 4 *Depending on the payoff parameters B_0, B_1, B_2:*

- *If $2B_1 \geq B_2 + B_0$, in equilibrium, the network behaves as a shrinking network, which generates the socially optimal order of stopping.*
- *If $2B_1 \in (B_2 + \frac{B_2}{B_1}B_0, B_2 + B_0)$, in equilibrium, the network behaves as a shrinking network while the socially optimal order of stopping is regular fragmenting.*
- *If $2B_1 < B_2 + \frac{B_2}{B_1}B_0$, in equilibrium the network is characterized by random fragmenting while the socially optimal order of stopping is regular fragmenting.*

To illustrate the results, consider the case where the initial network is randomly drawn to be a line of size 5. If players behave as in the shrinking network case, the resulting aggregate payoff is $4B_1 + B_0$. If players behave as in the regular fragmenting networks, where the initial player to stop is constrained to be either player 2 or player 4, the resulting aggregate payoff is $2B_2 + 3B_0$. Player 2 by stopping first, increases the payoff of player 1 from B_1 to B_0, at a cost of $B_2 - B_1$. Under the condition $2B_1 < B_2 + B_0$, this increases welfare but requires player 2 to internalize this externality. In the case of the random fragmenting network, with 1 chance out of 3, player 3 exits first, leaving players 1 and 2 and players 4 and 5 in a shrinking network. In that sense, the random fragmenting network decreases the degree of fragmentation. If $2B_1 < B_2 + B_0$, this decreases welfare as shown in Proposition 4.[9]

4.2 Spatial inefficiency and market shakeout

We consider in this section the second new source of inefficiency in waiting games due to the neighborhood structure. We describe it for the application to the exit decision by firms, or industry shakeouts, one of the standard examples of war of attrition games, and derive further implications of our model. The network can be seen in this case as representing a particular spatial distribution of firms. In this application, the assumption that players do not observe the number of neighbors of their own neighbor could appear extreme. However, it is only meant to capture the idea that players are not aware of the full structure of the network.

Assume firms are currently making zero profits. If they exit the market while they still have a neighbor, they get a payoff B_i (they can sell the machinery for instance) and if they are the last firm standing amongst the set of initial neighbors, they make a benefit $B_0 > B_i \ \forall i > 0$, corresponding to the discounted flow of local monopoly profits. This fits exactly the framework considered in the previous sections. Furthermore, according to the results of section 3 since $\overline{\gamma}_2 > \overline{\gamma}_1 = 0$, in equilibrium, the network should behave as a fragmenting network.[10]

In this application, it is natural to think that the shape of the final distribution of firms can be of great significance. For instance, suppose that customers at a distance of more than one link from a firm cannot be profitably served by that firm given their transport cost. In that case, the socially optimal distribution of firms would be equally spaced firms separated by a gap of size 1, what is achieved by regular fragmenting. However, as described in Section 4.1, the equilibrium is characterized by *random* fragmenting and there is thus no reason for the final spatial distribution to be equally spaced. In this section we characterize how far the equilibrium is from the equally spaced distribution.

We start with the limit case such that initially the line is fully connected so that $p_2(0) = 1$.[11] As described in Proposition 3, the equilibrium is such that initially only

[9] Two extreme cases are of interest. In the case where $B_0 = B_1 > B_2$, the first case of Proposition 4 holds. In the case where $B_2 = B_1 < B_0$, the third case of Proposition 4 holds.

[10] In Appendix B2, we consider an alternative model, closer to the classic war of attrition, where only the last firm gets benefit B_0 and all firms have to pay a flow cost c while staying in. We show that this gives qualitatively equivalent results: initially types 2 mix, until a date where all of them have entered. The only difference is that the Bellman equation includes the cost of staying in and the date where types 1 start mixing is thus affected.

[11] We defined the model originally so that $p_2(0) < 1$, or equivalently $p_1(0) > 0$. This is relevant for the case of shrinking networks, because there a positive initial fraction of types $k = 1$ is needed for the

types 2 mix until a date t_2 is reached where only isolated pairs of firms are left. After t_2, the firms in each such pair play a classical war of attrition until one firm from each such pair exits. The last surviving firm from each pair gets payoff B_0.

To characterize the spatial inefficiency, we are interested in two elements. First, we determine the proportion of firms getting payoff B_0, fraction we denote by p_e. It is easy to find natural upper and lower bounds for p_e by considering two unlikely extreme outcomes of the game. Suppose that in the first phase of the game every second firm exits, i.e the order of exits is regular fragmenting. In this case no firm exits in the second phase (there are no pairs of firms left) and hence we get $p_e = 1/2$. Consider the other extreme where in the first phase only every third firm exits. In this case there are only pairs of players left in the second phase and of those every second firm is yet to exit. In the end, the overall fraction of firms that survive is $p_e = 1/3$ in such a case. Naturally, the true value of p_e lies somewhere between these cases.

Second, we characterize the random variable measuring the gap between two consecutive firms at the end of the game, a random variable that we denote l_g that takes values in $\{1, 2, 3\}$.[12] If firms were equally spaced, l_g would be degenerate at value 1 in the case where $p_e = 1/2$. We describe in the following result the actual distribution of l_g:

Proposition 5 *At the end of the game, the spatial distribution of firms is such that:*

1. *The proportion of remaining firms is* $p_e = \frac{1}{2}\left(1 - e^{-2}\right) \backsimeq 0.43$.
2. *The probability distribution*[13] *of* l_g*, the gap between two consecutive firms, is:*

$$
\begin{aligned}
P\left[l_g = 3\right] &= p^2 \frac{1}{4} \backsimeq 0.01 \\
P\left[l_g = 2\right] &= p^2 \frac{1}{2} + 2p(1-p)\frac{1}{2} \backsimeq 0.21 \\
P\left[l_g = 1\right] &= p^2 \frac{1}{4} + 2p(1-p)\frac{1}{2} + (1-p)^2 \backsimeq 0.78,
\end{aligned}
$$

where $p = 2\frac{1}{1+e^2}$.

equilibrium to exist. For the case of fragmenting networks, there is no problem in having $p_2(0) = 1$. Alternatively, one can view this as characterizing the limit of the equilibrium as p_2 is arbitrarily close to one.

[12] To see that maximum gap size is 3, note that after the first phase of the game there are only gaps of size 1. The maximal growth of an individual gap during the second phase occurs if there is a pair of players on both sides of the gap, and the player closer to the gap in each such pair exits in the second phase, resulting in a gap of size 3.

[13] This result relies on the Markovian assumption.

To get an intuition of what drives the spatial inefficiency, it is helpful to think about the problem of deriving p_e recursively.[14] Suppose that at some date t during the first phase of the game, there is some line segment of n consecutive firms that have not exited yet (i.e. the left-hand side neighbor of the 1^{st} firm and the right-hand side neighbor of the n^{th} firm in this segment have exited, but none of those n firms have). Let us denote by $\xi(n)$ the expected number of firms within this segment that will survive until the end of the game. We can derive a recursive formula for $\xi(n)$ by noting that the interior players $2, ..., n-1$ of this segment all exit with the same hazard rate, while the boundary firms 1 and n do not randomize. It follows that the identity of the next firm to exit, denote this firm k, is uniformly distributed across the interior firms. Once the next exit takes place, the segment splits into two shorter segments of lengths $k-1$ and $n-k$, and the expected number of surviving firms within those shorter segments are $\xi(k-1)$ and $\xi(n-k)$, respectively. Hence, we can write $\xi(n)$ recursively as

$$\begin{aligned}\xi(n) &= \sum_{k=2}^{n-1} \Pr(\text{firm } \; k \text{ is the next to exit}) \cdot [\xi(k-1) + \xi(n-k)] \\ &= \frac{1}{n-2} \sum_{k=2}^{n-1} [\xi(k-1) + \xi(n-k)] .\end{aligned}$$

This recursive relationship can be expressed as a difference equation for $\xi(n)$ by computing

$$\begin{aligned}& (n-2)\,\xi(n) - (n-3)\,\xi(n-1) \\ = \;& \frac{n-2}{n-2} \sum_{k=2}^{n-1} [\xi(k-1) + \xi(n-k)] - \frac{n-3}{n-3} \sum_{k=2}^{n-2} [\xi(k-1) + \xi(n-k-1)] \\ = \;& 2\xi(n-2) ,\end{aligned}$$

which gives

$$\xi(n) = \frac{n-3}{n-2}\xi(n-1) + \frac{2}{n-2}\xi(n-2) . \tag{6}$$

Noting that in line segments of length 1 and 2, exactly one player will stay forever, we get the initial condition $\xi(1) = \xi(2) = 1$.

With this initial condition the difference equation (6) pins down a unique sequence,

[14] We thank Juuso Välimäki for suggesting this line of thinking. We give an alternative method of proof in the appendix.

where an arbitrary element can be expressed in closed form as:

$$\xi(n) = \frac{1}{2}\left(1 + n - \frac{(-2)^n + (n+1)! \sum_{k=0}^{n+1} \frac{(-2)^k}{k!}}{n!}\right).$$

Taking the limit, we obtain the expression:

$$p_e = \lim_{n\to\infty} \frac{\xi(n)}{n} = \frac{1}{2}\left(1 - e^{-2}\right) \backsimeq 0.43.$$

Proposition 5 describes precisely how the final distribution of firms differs from an equally spaced distribution. Overall, approximatively 43 percent of firms remain at the end of the game. This implies that at least some firms are separated by a gap of more than 2. In fact we find that 28 percent of firms are in this situation, while gaps of 3 are rather rare.

5 Timing inefficiency and technology adoption in networks

The timing inefficiency, standard in waiting games, is also present when we introduce the neighborhood structure. We describe it here in the context of the adoption of new technologies by firms organized in a line. The action "stop" represents here adopt the technology. When a firm adopts, it decreases its neighbors' cost of adoption through either technological spillovers or informational spillovers. In this context, we study the effect of subsidy programs aimed at speeding up adoption. Many countries have in place large scale subsidy programs to support adoption of technologies. This includes subsidies for agricultural techniques (such as fertilizers) in developing countries, health saving technologies, or environmentally friendly technologies in developed countries. While a variety of reasons can justify such subsidy programs, we highlight in this paper the role of subsidies to correct the inefficiencies linked to coordination failures.[15]

[15] Of course, different motivations drive public intervention in these different areas. The main justification for subsidies in the case of environmentally friendly technologies, and to some extent health related products, is the internalization of an externality. For agricultural techniques, as reported in Duflo et al. (2011), there is much less consensus on the source of market failure justifying state intervention. Some cite informational problems while others invoke behavioral biases.

5.1 Application of the model to adoption of technologies

We first examine how our model applies to adoption of technologies and argue that shrinking networks are more relevant in this case.

Consider a situation where spillovers between neighbors are technological, so that a link represents technological proximity between two members of the network.[16] Upon adoption, the adopting firm trains employees and potentially trains suppliers if the new technology affects the interactions with suppliers. We know that there is large mobility of skilled labor across firms in the same technological areas and that firms situated close to each other often share suppliers. Both effects imply that adoption by one firm may reduce the adoption costs of its neighbors (Jaffe et. al. 1993, Almeida and Kogut 1999).

In this application, a player is characterized by two state variables: a, the number of active neighbors (those who have not yet adopted) and i, the number of inactive neighbors (those who adopted in the past and provided spillovers). The number of inactive neighbors determines the payoff when stopping while the number of active neighbors impacts the incentives to wait. However, in our model, we allow for a single state variable k, the number of neighbors. The results we obtained directly apply to the context of technology adoption as described above if we assume that all players start out with the same number of neighbors, i.e $a + i = N$. In this case, keeping track of the number of active neighbors is sufficient, since $i = N - a$. We show in Appendix B3 that the equilibrium structure that we identify in Section 3 is preserved if we do not impose the restriction $a + i = N$ and consider the general case with two state variables. The main difference lies in the transitions from one state to another when a neighbor adopts, i.e. moves from being active to inactive.

Consider as an illustration the following specification of payoffs. Suppose the time invariant benefit of adopting the technology is given by B and denote c_a the cost of adoption for a player who does not benefit from spillovers. The adoption by one neighbor reduces the cost by a factor σ_1, and the next adoption reduces the cost further by another factor σ_2. The benefit of adoption for a player with i inactive neighbors (who have already exited) is thus given by $B - \left(\prod_{l=1}^{i} \sigma_l\right) c_a$. Calling k the number of active neighbors, and imposing the restriction $i + k = 2$, payoffs can be parameterized by k in the following way: $B_2 = B - c_a$, $B_1 = B - \sigma_1 c_a$ and $B_0 = B - \sigma_1 \sigma_2 c_a$, since having no active neighbors directly implies that there are 2 inactive neighbors who already provided spillovers.

Section 3 established that in equilibrium, the network behaves as a shrinking network

[16] Informational spillovers, due for instance to the fact that firms can observe the adoption techniques used by their neighbors, are formalized in Appendix B1.

if and only if $\overline{\gamma}_1 > \overline{\gamma}_2$. Using the parametrization of profits introduced above, we conclude that the condition is satisfied under fairly general conditions on σ_l. This is the case for instance if σ_l is constant, since then $\overline{\gamma}_1 = \frac{B-\sigma c_a}{\sigma(1-\sigma)c_a} > \overline{\gamma}_2 = \frac{B-c_a}{2(1-\sigma)c_a}$. To be in the case of the fragmenting network requires σ_1 to be very small relative to σ_2, which appears unlikely.[17] The shrinking network case also seems to be the most relevant case empirically: speed of technology diffusion is often described using measures of distance covered by year (see the survey by Geroski 2000). We thus restrict attention in this section to shrinking network and we study the effect of subsidy programs aimed at speeding up adoption.

5.2 Permanent subsidy

We start with the case where a fixed subsidy $s > 0$ is given to any player at the time of adoption. Typically there is a deadweight loss of funds raised to finance such subsidy programs. To calculate overall welfare we thus assume throughout the analysis of the different subsidy programs that, for a given subsidy s awarded, the social cost is given by $(1+\alpha)s$, where parameter $\alpha > 0$ measures the welfare loss associated with raising and transferring funds. We also denote by $G^{pe}(s)$ the expected payoff of an arbitrary player given subsidy s and by $C^{pe}(s)$ the expected discounted subsidy payment to an arbitrary player. The total welfare with permanent subsidy s, denoted $W^{pe}(s)$, is then

$$W^{pe}(s) = G^{pe}(s) - (1+\alpha) C^{pe}(s). \tag{7}$$

From the players' perspective a permanent subsidy just amounts to replacing payoff terms B_k with $B_k + s$, $k = 0, 1, 2$. We showed in section 3.3 that for shrinking networks, the belief that the neighbor is of type 1, the mixing rate of a type 1 player and the expected entry rate of a random neighbor remain constant throughout the game. In particular the hazard rate of adoption by an arbitrary neighbor is given by:

$$\gamma^{pe}(s) = p_1(t) \lambda_1^{pe}(s) = \frac{r(B_1 + s)}{B_0 - B_1}. \tag{8}$$

This is linearly increasing in s so we see that the subsidy speeds up adoption.

We first compute the welfare gain from the policy $G^{pe}(s)$. Type $k = 0$ players adopt immediately and get $B_0 + s$. Type $k = 1$ players are indifferent between adopting immediately and waiting, and hence their payoff is $B_1 + s$. Types $k = 2$ strictly prefer

[17] Could be the case if spillovers come from suppliers and a sufficient mass of firms needs to adopt to give incentives for the supplier to also invest in the new technology.

to wait until one of their neighbors adopts, at which point their payoff becomes $B_1 + s$. Since each of her two neighbors adopt with rate $\gamma^{pe}(s)$, we can compute the payoff of type $k = 2$ as

$$\begin{aligned} V_2^{pe}(s) &= \int_0^\infty 2\gamma^{pe}(s)\, e^{-2\gamma^{pe}(s)t} e^{-rt} (B_1 + s)\, dt = \frac{2\gamma^{pe}(s)}{2\gamma^{pe}(s) + r} (B_1 + s) \\ &= \frac{2(B_1 + s)^2}{B_0 + B_1 + 2s}. \end{aligned}$$

The expected payoff of an arbitrary players is then:

$$G^{pe}(s) = q_0 (B_0 + s) + q_1 (B_1 + s) + q_2 \frac{2(B_1 + s)^2}{B_0 + B_1 + 2s}. \tag{9}$$

We derive in the Appendix the expected subsidy payment $C^{pe}(s)$ in a similar way, and obtain the following result:

Proposition 6 *A permanent subsidy s induces types 1 to adopt with constant hazard rate*

$$\lambda_1(s) = \frac{1}{p_1(0)} \frac{r(B_1 + s)}{B_0 - B_1}.$$

Furthermore, there exists $\alpha^* > 0$ such that if $\alpha < \alpha^*$, a permanent subsidy set at some positive level $s > 0$ strictly increases total welfare. If $\alpha \geq \alpha^*$, no permanent subsidy increases total welfare.

The proposition shows that if the social cost of subsidies is not too high (i.e. α is not too large), then a permanent subsidy increases total welfare. The intuition is that by increasing incentives to adopt, the subsidy speeds up adoption. Every adoption decision entails a positive externality on all later adopters and thereby the positive welfare effect of the subsidy propagates through the network. Overall, socially wasteful delays are diminished.

However, there are more powerful ways to accelerate adoption. Intuitively, if the expected subsidy payment tomorrow is lower than today, then there is an additional incentive to adopt today rather than tomorrow. We next turn to time varying policies that exploit this logic.

5.3 Expiring subsidy

Suppose that a subsidy is set at level $s > 0$, but expires at some random time that is exponentially distributed with parameter κ. This could capture the idea that political turnover or even economic shocks, can lead at any time to the interruption of subsidy programs. Any player that adopts before expiration gets payoff $B_k + s$ (where k is her type), while a player that adopts after expiration gets B_k.

We derive an equilibrium, where type $k = 1$ stops at a constant rate $\lambda_1^{ra}(s,\kappa)$ until expiration (after expiration, the game continues as the original game without any subsidies). This adoption rate induces an arbitrary neighbor to adopt at rate:

$$\gamma^{ra}(s,\kappa) = p_1(0)\,\lambda_1^{ra}(s,\kappa).$$

To derive the equilibrium value of $\gamma^{ra}(s,\kappa)$, note that type $k = 1$ must be indifferent between stopping at t and $t + dt$:

$$\begin{aligned} B_1 + s &= \gamma^{ra}(s,\kappa)\,dt\,(1 - rdt)\,(B_0 + s) + \kappa dt\,(1 - rdt)\,B_1 \\ &\quad + (1 - \gamma^{ra}(s,\kappa)\,dt)\,(1 - \kappa dt)\,(1 - rdt)\,(B_1 + s). \end{aligned}$$

The equilibrium exit rate of an arbitrary neighbor is thus given by:

$$\gamma^{ra}(s,\kappa) = \frac{r(B_1 + s) + \kappa s}{B_0 - B_1}.$$

We see from this equation that increasing κ induces a higher rate of adoption.

We compute in the Appendix the total welfare of this policy and show that it is strictly increasing in κ. This implies in particular that the highest welfare is obtained in the limit $\kappa \to \infty$ and the lowest level is obtained when $\kappa \to 0$, where the welfare converges to the level obtained with the permanent subsidy.

Proposition 7 *A subsidy set at level $s > 0$ that expires at rate $\kappa > 0$ induces a higher total welfare than a permanent subsidy at level s. Moreover, the total welfare is strictly increasing in κ.*

5.4 Smoothly declining subsidy

The planner can further induce early adoption by letting the subsidy level $s(t)$ decrease over time. Such a policy induces type $k = 1$ to adopt with some time varying hazard rate. We look for an equilibrium, where type $k = 1$ is indifferent at all times between adopting and waiting, and adopts with a time varying hazard rate $\lambda_1^{sm}(t)$. As before, p_1 and p_2 remain constant at their initial values $p_1(0)$ and $p_2(0)$ throughout, and the resulting hazard rate of adoption by an arbitrary neighbor can be written as

$$\gamma^{sm}(t) = p_1(0)\,\lambda_1^{sm}(t).$$

The indifference condition for type 1 between adopting at t and $t + dt$ is:

$$B_1 + s(t) = \gamma^{sm}(t)\,dt\,(B_0 + s(t)) + (1 - \gamma^{sm}(t)\,dt)(1 - rdt)\left(B_1 + s(t) + \dot{s}(t)\,dt\right).$$

This gives:

$$\gamma^{sm}(t) = \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1}.$$

We see from this equation that the decline in the subsidy path ($\dot{s}(t) < 0$) increases the rate of adoption as expected.

We show in Proposition 8 that there exists a declining subsidy that achieves a strictly higher total welfare than any permanent or randomly expiring policy. Intuitively, a decline in the subsidy boosts adoption similarly to the threat of expiration. Unlike randomly expiring policy, though, it has the additional benefit that the realized subsidy payments keep getting smaller as time goes by, thus diminishing the realized welfare cost of subsidies.[18]

Proposition 8 *There exists a smoothly declining subsidy that gives a strictly higher total welfare than any permanent or any randomly expiring subsidy.*[19]

Figure 1 compares the total welfare of the three different policies as a function of initial subsidy level s, where the randomly expiring policy is in the limit $\kappa \to \infty$, and the smoothly declining policy is in the limit, where the decline is infinitely fast. Note that the optimal level of s is different in each policy.[20]

5.5 Reward when a neighbor adopts

So far we have considered uniform subsidy policies, where a given subsidy is paid to any player that adopts at a given time. The reason for focusing on such policies is the

[18] The proof of the following result is in the Appendix. We analyze there a subsidy that declines linearly from initial level s to zero, and take the limit where the decline path is infinitely steep. It turns out that for any given initial level s this policy is strictly better than the other policies we have considered for that same level s. The result then follows from a simple continuity argument.

[19] Unless α is very large, in which case no subsidy policy can increase total welfare.

[20] We do not have a formal result characterizing the optimal subsidy policy amongst all possible time varying policies. However, we conjecture that the best policy is indeed the one where the decline path is made as steep as possible.

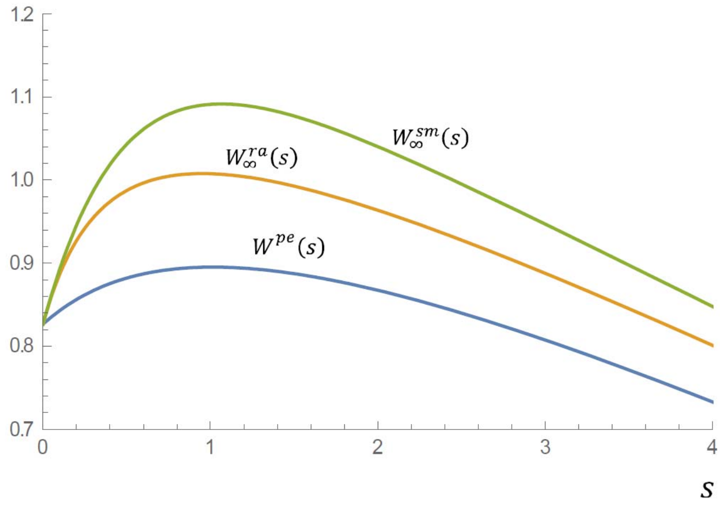

Figure 1: Comparing subsidy programs. Parameter values are: $B_0 = 2$, $B_1 = 1$, $\alpha = 0.1$, $q_0 = 0.04$, $q_1 = 0.32$, $q_2 = 0.64$

implicit assumption that the regulator does not have access to detailed neighborhood information. In some situations it may however be possible, for example by relying on the agents' self reporting, to base subsidy payments on the neighbors' adoption decisions. To demonstrate the potential of such policies, we consider a neighbor reward policy, where a fixed reward $m > 0$ is given to any player at the time the first of her neighbors adopts after she has adopted herself. Denoting by $G^{na}(m)$ the expected payoff of an arbitrary player given reward m and by $C^{na}(m)$ the expected discounted subsidy payment to an arbitrary player, the total welfare with reward m is

$$W^{na}(m) = G^{na}(m) - (1+\alpha) C^{na}(m) \tag{10}$$

When $B_0 - B_1 \le m \le B_0 - B_2$, there is an equilibrium where agents of type 1 adopt immediately, triggering an instantaneous cascade of adoptions. All levels of reward within this range are dominated by $\overline{m} = B_0 - B_1$, which achieves the same adoption pattern than the higher levels at a lower cost. This level of reward achieves the total welfare

$$W^{na}(\overline{m}) = B_0 - (1+\alpha)\left(q_2 + \frac{q_1}{2}\right)(B_0 - B_1).$$

For a reward $m \le \overline{m}$, we look for an equilibrium as in the shrinking networks game, where the belief p_1 that the neighbor is of type 1, the mixing rate $\lambda_1^{na}(m)$ of a type 1 player and the expected entry rate of a random neighbor $\gamma^{na}(m) = p_1(0)\lambda_1^{na}(m)$ remain constant throughout the game. In that case, the policy is equivalent, from the point of view of the agents, to replacing payoff terms B_1 with $B_1 + \frac{\lambda_1^{na}m}{\lambda_1^{na}+r}$, $k = 0, 1, 2$. In particular the hazard rate of adoption by an arbitrary neighbor must then satisfy

$$\gamma^{na} = \frac{r\left(B_1 + \frac{\lambda_1^{na}m}{\lambda_1^{na}+r}\right)}{B_0 - \left(B_1 + \frac{\lambda_1^{na}m}{\lambda_1^{na}+r}\right)} = \frac{r\left(B_1 + \frac{\gamma^{na}m}{\gamma^{na}+rp_1(0)}\right)}{B_0 - \left(B_1 + \frac{\gamma^{na}m}{\gamma^{na}+rp_1(0)}\right)}.$$

Rearranging this expression, we see that for any $m \in [0, \overline{m})$, the equilibrium adoption rate of a neighbor γ^{na} is the unique positive solution of the equation

$$\gamma^{na}\left((B_0 - B_1)(\gamma^{na} + rp_1) - \gamma^{na}m\right) - r\left(B_1(\gamma^{na} + rp_1) + \gamma^{na}m\right) = 0.$$

The left-hand side is increasing in γ^{na} for $\gamma^{na} > 0$ and decreasing in m, so it follows that the function $\gamma^{na}(m)$ is increasing in m. For $m = 0$, $\gamma^{na}(m) = \frac{rB_1}{B_0 - B_1}$ and as m approaches $\overline{m}$, $\gamma^{na}(m)$ goes to infinity. From the point of view of both the agents and the planner, the policy m is equivalent in terms of payoff to a discriminating permanent subsidy $s^* = \frac{\gamma^{na}m}{\gamma^{na}+rp_1(0)}$ that is given on the date of adoption exclusively to adopting type 1 agents, not to type 0 agents. As we show formally in the proof of the following result, such a discriminating subsidy policy can provide adoption incentives to type 1 agents at a lower cost than a non-discriminating subsidy policy. We thus have:

Proposition 9 *For any permanent subsidy $s > 0$, there exists a neighbor reward $m \in [0, \min\{B_0 - B_1, s\}]$ that yields a greater social welfare than the permanent subsidy s. As a consequence the optimal neighbor reward program is superior to the optimal permanent subsidy policy.*

6 Conclusion

In this paper we have studied a waiting game with a network structure, highlighting the application to the adoption decisions of firms that can benefit from positive spillovers due to adoption of neighbors. The neighbourhood structure gives rise to two additional inefficiencies on top of the timing inefficiency standard in waiting games: an inefficiency in the order of exit and an inefficiency due to the final distribution of nodes at the end of the game. Both are due to the fact that players do not internalize the positive externalities they can impose on their neighbors by adopting first.

In order to highlight the special dynamics of stopping and the new sources of inefficiencies due to the neighborhood structure, we focus in this paper our attention to the simplest possible network structure, the line. The extension to larger networks will be the object of future work. Preliminary results suggest that, in larger networks with no cycles, the dynamics of stopping are qualitatively similar in the case of fragmenting networks, where the large network fragments into smaller ones over time. The analysis of shrinking networks turns out to involve additional modeling challenges.

First, larger networks could have *cascades*. Consider such a large network and suppose that the equilibrium corresponds to our shrinking network, where type k randomizes while types above k wait. Whenever a player becomes type $k-1$ as a result of one of her neighbors stopping, her incentive to stop instantly increases, and she wants to stop immediately. Every stopping decision starts a chain-reaction: some neighbors of the stopping player may become type $k-1$, immediately stop, and thus spread the cascade further. If the number of connections in the network is high, those cascades may eventually become very long and this will bring into question the existence of an equilibrium similar to the one that we analyze.

Second, real networks often have cycles and hubs. Adding such features in the model will be challenging, since it requires modifying our modeling approach, where every player believes the type of each of her neighbors to be identically and independently distributed. Addressing such issues is left for future work.

REFERENCES

Ali, N. and D. Miller. 2013. "Enforcing Cooperation in Networked Societies" *working paper.*

Almeida, P. and B. Kogut. 1999. "Localisation of knowledge and the mobility of engineers in regional networks." *Management Science*, 45, 905-917.

Atkin D., A. Chaudhry, S. Chaudry, A. K. Khandelwal and E. Verhoogen. 2017. "Organizational barriers to technology adoption: evidence from soccer-ball producers in Pakistan " *The Quarterly Journal of Economics*, 132(3), 11011164.

Bandiera, O., and I. Rasul. 2006. "Social Networks and Technology Adoption in Northern Mozambique." *Economic Journal*, 116(514), 869902.

Bulow, J. and P. Klemperer. 1999. "The Generalized War of Attrition." *American Economic Review*, 89(1), 175189.

Conley, T. G., and C. R. Udry. 2010. "Learning about a New Technology: Pineapple in Ghana." *American Economic Review*, 100(1), 35-69.

Duflo, E., M. Kremer, and J. Robinson. 2011. "Nudging Farmers to Use Fertilizer: Theory and Experimental Evidence from Kenya." *American Economic Review*, 101(6), 2350-90.

Foster, A. D., and M. R. Rosenzweig. 1995. "Learning by Doing and Learning from Others: Human Capital and Technical Change in Agriculture," *Journal of Political Economy* , 103(6), 1176-1209.

Fudenberg, D. and J. Tirole. 1986. "A theory of exit in duopoly," *Econometrica*, 54(4), 943-960.

Galeotti A., Goyal S., Jackson M.O., Vega-Redondo F. and L. Yariv. 2010, "Network games," *Review of Economic Studies* 77, 218-244.

Geroski, P. A., 2000, "Models of technology diffusion," *Research Policy*, 29(4-5), 603-625.

Goyal, S. 2007. "Connections: An Introduction to the Economics of Networks," *Princeton University Press.*

Jackson, M.O. and Y. Zenou. 2014. "Games on networks " *Handbook of Game Theory, Vol. 4, Peyton Young and Shamuel Zamir, eds., Elsevier Science.*

Jackson, M.O., T. Rodriguez-Barraquer, and X. Tan. 2012. "Social Capital and Social Quilts: Network Patterns of Favor Exchange," *American Economic Review* 102, 1857- 1897.

Jackson M.O. and L. Yariv. 2005. "Diffusion on Social Networks," *Economie Publique* 16, 3-16.

Jackson M.O. and L. Yariv. 2007. "The diffusion of behavior and equilibrium structure properties on social networks," *American Economic Review Papers and Proceedings* 97, 92-98.

Jaffe, A. B., Trajtenberg, M. and R.Henderson. 1993. "Geographic localisation of knowledge spillovers as evidenced by patent citations." *Quarterly Journal of Economics*, 108(3), 577-598.

Kennan, J. and R. Wilson. 1989. "Strategic bargaining models and interpretation of strike dates." *Journal of Applied Econometrics*, 4, 87-130.

Leduc, M. and Jackson, M. and R. Johari. 2017. "Pricing and Referrals in Diffusion on Networks." *Games and Economic Behavior*, 104, 568-594.

Maynard Smith, J. 1974. "The theory of games and the evolution of animal conflicts." *Journal of Theoretical Biology*, 47(1), 209-221.

Raub, W. and J. Weesie. 1990. "Reputation and efficiency in social interactions: An example of network effects" *American Journal of Sociology* 96, 626-654.

World Bank. 2007. "World Development Report 2008: Agriculture For Development."

7 Appendix A

Proof of Lemma 1

As explained in the main text, $P1$ and $P2$ hold at $t = 0$.

We first prove that $P2$ is satisfied at all histories on path. Consider player i and a history at date t, on path for this player, at which i has not stopped. The type at date t of each neighbor j of i is affected only by the actions of the set $N_s^{i,j}$ of agents (excluding i, but including j) that are indirectly connected to i through j at dates $s \in [0, t]$, (and of course by the strategy of player i). For any two distinct neighbors j and j' of i, and for all s, we have $N_s^{i,j} \cap N_s^{i,j} = \emptyset$. Thus the strategies of agents in $N_t^{i,j}$ are independent across all neighbors j, conditionally on i having not stopped until date s, because the strategies of the players are not correlated. Since player i updates his beliefs according to Bayes' rule, and since player i believes at date 0 that his neighbors' types are independently distributed, it follows that at date t, player i still believes that the types of her neighbors are independently distributed. This proves that $P2$ is satisfied at any history on path for player i.

We now prove that $P1$ is satisfied at all histories on path. Let i and i' be two distinct agents, with neighbors j and j'. For all $s \in [0, t]$, let $G_s^{i,j}$ the set of links of the line that links the agents in $N_s^{i,j}$ and let $G_s^{i',j'}$ be the analogous object for agents in $N_s^{i',j'}$. Because conditions $P1$ and $P2$ hold at date 0, and because the beliefs of the agents at that date are identical, it follows that the belief of player i about the structure of $G_0^{i,j}$ is identical to the belief of player i' about the structure of $G_0^{i',j'}$. That is, these two agents have identical beliefs about the sequence of links, ignoring the labels of the agents in those structures. Then, because $G_t^{i,j}$ (respectively $G_s^{i',j'}$) is only affected by the actions of the players in the line $\left(N_s^{i,j}, G_s^{i,j}\right)$ (respectively in the line $\left(N_s^{i',j'}, G_s^{i',j'}\right)$) and that these lines are identically distributed at all date $s < t$ and all player's strategies are independent, it follows that $\left(N_t^{i,j}, G_t^{i,j}\right)$ and $\left(N_t^{i',j'}, G_t^{i',j'}\right)$ are identically distributed as well.

Proof of Lemma 2

Consider an arbitrary symmetric equilibrium σ and let F be the distribution of stopping dates of an arbitrary neighbor of an arbitrary player i conditional on i never stopping. Let $T \in \mathbb{R} \cup \{+\infty\}$ be the least upper bound of the support of F. We prove the lemma through the following steps:

1. The cdf F has no atoms. By contradiction, suppose that F has an atom at date t. Then consider a realization of the equilibrium outcome, where player i of type $k \geq 1$

stops exactly at date t. Since F has an atom at t, i expects each of his neighbors to stop at date t with positive probability. It is then a profitable deviation for her not to stop at date t, but rather at date $t + dt$, as the waiting cost of doing so is proportional to dt whereas the expected gain of delaying is bounded from below by $(F(t^{+}) - F(t^{-}))(B_{k-1} - B_k) > 0$. A contradiction.

2. There is no interval of positive length included in $[0, T]$ over which F is constant. By contradiction, suppose that $\underline{t}$ and $\overline{t}$ are such that $0 \leq \underline{t} < \overline{t} < +\infty$, and F is constant on $\left[\underline{t}, \overline{t}\right]$, but not on a right-neighborhood of $\overline{t}$. This implies that there is at least one type k such that $\lambda_k(t) > 0$ in a right neighborhood of $\overline{t}$. Consider a realization of the equilibrium outcome, and suppose that player i of type k with at least one neighbor stops at date $\overline{t} + \varepsilon$, where ε is arbitrarily small. Then stopping instead at date $\underline{t}$ (at which the type of i was the same) would have been a profitable deviation for player i. Evaluated at date 0, the cost of doing such a change is approximately equal to $\gamma(t)(B_{k+1} - B_k) e^{-r(t+\varepsilon)}\varepsilon$, which is small as ε goes to 0. Meanwhile, the benefit of doing so is bounded below by $B_{k+1}e^{-r\left(\varepsilon + \overline{t} - \underline{t}\right)} > 0$,which exceeds the cost. A contradiction.

3. It must be that $T = +\infty$. To se this, suppose by contradiction that $T < +\infty$. In a realization of the outcome, suppose that an player i stops at date $T - \varepsilon$. Since she knows that all remaining neighbors will exist between $T - \varepsilon$ and T, it is well worth delaying stopping after date T. By doing so, player i strictly gains by stopping ε periods later, the cost of which is bounded above by $e^{-rt}B_k\varepsilon$ and the benefit of which is bounded below by $e^{-rT}(B_{k-1} - B_k) > 0$. Thus it must be that $T = +\infty$.

Proof of Proposition 1

We prove the result in the case $\mu_1 > \mu_2$. The other case is perfectly symmetric.

In a symmetric equilibrium, each player faces the same probability distribution for the other player's stopping time, which we denote by $F(t)$, as in the main model. By the same arguments as in Lemma 2, the distribution has no atoms, and its support is $[0, \infty)$. We may hence describe the equilibrium by hazard rates of the two types, $\lambda_1(t)$ and $\lambda_2(t)$, where at least one of them is non-zero for each $t \geq 0$. Letting $p_1(t)$ denote the posterior probability at t that the other player is of type 1, we have

$$\frac{f(\tau)}{(1 - F(\tau))} = p_1(t)\lambda_1(t) + (1 - p_1(t))\lambda_2(t).$$

The expected payoff for type j of the strategy "stop at time τ if the other player has not yet stopped" is given by:

$$W_j(\tau) = \left[\int_0^{\tau} e^{-rt} \left(B_{j-1} \right) f(t)dt + (1 - F(\tau))e^{-r\tau}(B_j) \right].$$

Differentiating this, we have

$$\frac{dW_j\left(\tau\right)}{d\tau} = e^{-r\tau}B_{j-1}f(\tau) - f(\tau)e^{-r\tau}B_j - r(1 - F(\tau))e^{-r\tau}B_j,$$

and it follows that

$$\frac{dW_j\left(\tau\right)}{d\tau} > (=) \, (<) \, 0 \Longleftrightarrow \frac{f(\tau)}{(1 - F(\tau))} > (=) \, (<) \, r\frac{B_j}{B_{j-1} - B_j} \equiv \mu_j.$$

For a player of type j to mix in an interval $[t, t']$, he must be indifferent between stopping at any date $\tau \in [t, t']$, and we must have:

$$\frac{dW_j\left(\tau\right)}{d\tau} = 0 \Longleftrightarrow \frac{f(\tau)}{(1 - F(\tau))} = \mu_j.$$

Since $\mu_1 \neq \mu_2$, only one type can be mixing in an interval. Suppose that type j mixes in some interval $[0, t']$ and the other type $k \neq j$ is willing to delay. Then we must have

$$\frac{dW_k\left(\tau\right)}{d\tau} \geq 0 \Longleftrightarrow \frac{f(\tau)}{(1 - F(\tau))} \geq \mu_k$$

for $\tau \in [0, t']$. Since $\mu_1 > \mu_2$, it must be that the type mixing initially is type $j = 1$ and the type that is waiting is $k = 2$, and so

$$\frac{f(\tau)}{(1 - F(\tau))} = p_1(t)\lambda_1(t) = \mu_1.$$

The updated belief that the other player is of type 1 is then given by Baye's rule:

$$p_1(t + dt) = \frac{p_1(t)(1 - \lambda_1(t)dt)}{p_1(t)(1 - \lambda dt) + (1 - p_1(t))}$$

So that

$$\frac{p_1(t + dt) - p_1(t)}{dt} = \frac{1}{dt}\frac{p_1(t)(1 - \lambda_1(t)dt) - p_1(t)(p_1(t)(1 - \lambda_1(t)dt) + (1 - p_1(t)))}{p_1(t)(1 - \lambda_1(t)dt) + (1 - p_1(t))}$$

Taking limits we have:

$$\dot{p}_1(t) = -\lambda_1(t)p_1(t)(1 - p_1(t)).$$

Since $\mu_1 = p_1(t)\lambda_1(t)$, we can reexpress as:

$$\dot{p}_1(t) = -\mu_1(1 - p_1(t)).$$

The solution of this differential equation is:

$$1 - p_1(t) = (1 - p_1(0))e^{-\mu_1 t},$$

and so $p_1(t)$ is strictly decreasing over time. Thus there exists a time t_1^b such that $p_1(t_1^b) = 0$, which we can solve as $t_1^b = -\frac{ln(1-p_1(0))}{\mu_1}$. After that date only players of type 2 are left, and the continuation game is a standard complete information waiting game with the unique symmetric equilibrium where the players mix at constant rate $\lambda_2(t) = \mu_2$.

Proof of Proposition 2

By the same argument as in the proof of Proposition 1, there must be some initial phase $[0, t']$ during which type 1 randomizes and type 2 waits, and where

$$\gamma(t) = \lambda_1(t)p_1(t) = \overline{\gamma}_1.$$

We next establish the result that as long as type 1 randomizes and type 2 waits, $p_1(t)$ remains constant, and so the initial phase never ends. Suppose that type 1 stops at rate $\lambda_1(t)$ and type 2 waits so that $\lambda_2(t) = 0$. We define two events:

- NE (no entry) the event that no entry takes place in the interval $[t, t + \epsilon]$.
- CS (change state) the event that the neighbor changes state during the interval $[t, t + \epsilon]$, which can only mean that his other neighbor stopped, i.e he moved from being a type 2 to a type 1.

Using these notations, we have:

$$\begin{aligned} p_1(t+\epsilon) &= \frac{P[k=2 \cap NE \cap CS]}{P[NE]} + \frac{P[k=1 \cap NE \cap CS^C]}{P[NE]} \\ &= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\gamma_1(t)\epsilon}{P[NE]} + \frac{P[NE|k=1 \cap SC^C]P[k=1 \cap SC^C]}{P[NE|k=1]p_1(t) + (1-p_1(t))} \end{aligned}$$

We now examine:

$$
\begin{aligned}
\frac{p_1(t+\epsilon)-p_1(\epsilon)}{\epsilon} &= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\gamma_1\,(t)}{P[NE]}+\frac{1}{\epsilon}\frac{p_1(t)(1-\lambda_1(t)\epsilon)-p_1(t)(p_1(t)(1-\lambda_1(t)\epsilon)+(1-p_1(t)))}{P[NE]} \\
&= \frac{p_2(t)(1-\lambda\,(t)\,\epsilon)\gamma_1\,(t)}{P[NE]}+\frac{1}{\epsilon}\frac{p_1(t)(1-p_1(t))\lambda\,(t)\,\epsilon}{P[NE]} \\
&= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\gamma_1\,(t)}{P[NE]}+\frac{p_1(t)(1-p_1(t))\lambda_1(t)}{P[NE]}.
\end{aligned}
$$

Taking the limit when ϵ goes to zero, $P[NE]$ converges to one and so

$$\dot{p}_1(t)=\gamma_1\,(t)\,(1-p_1(t))-\lambda\,(t)\,p_1(t)(1-p_1(t)).$$

Finally, by definition, $\gamma_1\,(t)=\lambda_1\,(t)\,p_1\,(t)$, so that

$$\dot{p}_1\,(t)=0.$$

Given that $p_1(t)$ and $\gamma_1\,(t)=\overline{\gamma}_1$ do not depend on time, the rate of mixing of types 1, $\lambda_1(t)$, also remains constant and is equal to $\lambda_1=\frac{\overline{\gamma}_1}{p_1(0)}$. This establishes the first part of the proposition.

We next derive the average time before stopping of a random member of the network. If the player is of type 0 (probability q_0), she enters immediately. If she is of type 1 (probability q_1), her stopping rate is $\lambda_1+\overline{\gamma}_1$, since she stops either because of her own mixing or because a neighbor stops. Finally, if she is of type 2, she first needs to transition to being a type 1, which occurs at a rate $2\overline{\gamma}_1$, then follows the same dynamic as a type 1. Overall the expected waiting time is given by:

$$
\begin{aligned}
E\,[T] &= q_0 0+q_1\frac{1}{\lambda_1+\overline{\gamma}_1}+q_2\left[\frac{1}{2\overline{\gamma}_1}+\frac{1}{\lambda_1+\overline{\gamma}_1}\right] \\
&= q_2\frac{1}{2\overline{\gamma}_1}+(q_1+q_2)\frac{1}{\lambda_1+\overline{\gamma}_1}.
\end{aligned}
$$

We showed above that $\lambda_1=\frac{\overline{\gamma}_1}{p_1(0)}$. Furthermore, we showed in Section 2.2 that

$$p_1\,(0)=\frac{q_1}{q_1+2q_2}.$$

Replacing these in the expression for $E\,[T]$, we get:

$$E\,[T]=(q_1+q_2)\,\frac{1}{2\overline{\gamma}_1}.$$

Being inversely proportional to $\overline{\gamma}_1$, $E[T]$ is increasing in B_0, decreasing in B_1 and independent of B_2. Furthermore, since $\overline{\gamma}_1$ is independent of q_1 and q_2, $E[T]$ is overall increasing in $q_1 + q_2$.

Proof Proposition 3

Following the same arguments as in Proposition 2, we can establish that while players of type 2 are mixing, the beliefs evolve according to:

$$\dot{p}_2(t) = -\gamma(t)p_2(t) - \lambda_2(t)p_2(t)(1 - p_2(t)).$$

Given that $\gamma(t) = \lambda_2(t)p_2(t)$, we obtain that

$$\dot{p}_2(t) = -\gamma(t),$$

i.e

$$p_2(t) = p_2(0) - \int_0^t \gamma(s)\, ds.$$

We see that $p_2(t)$ is a strictly decreasing function with derivative bounded away from zero, so there is a date t_2 such that $p_2(t_2) = 0$.

As we argued in the main text, we have

$$\gamma(t) = \frac{rB_2}{2(V_1(t) - B_2)},$$

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma(t) B_0 dt + (1 - \gamma(t)dt)(1 - rdt)\left(V_1(t) + \dot{V}_1(t)\, dt\right).$$

Using the value of $\gamma(t)$, we obtain:

$$\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0. \tag{11}$$

To establish the last result, we compare the values of $\dot{p}_2(t)$ in the two cases. Here we have:

$$\dot{p}_2(t) = -\frac{rB_2}{2(V_1(t) - B_2)}.$$

In the benchmark case we had:

$$\dot{p}_2(t) = -(1 - p_2(t))\frac{rB_2}{(B_1(t) - B_2)}.$$

Given that $V_1(t) \geq B_1$ and $p_2(t) \leq p_2(0) < \frac{1}{2}$, the posterior probability decreases faster in the benchmark case, so that $t_2 > t_2^b$.

Proof of Proposition 4

Take an arbitrary line segment of n players and consider all feasible orders in which those n players can stop. In all the arguments, the players are numbered from 1 to n from left to right. Denote by n_k, $k = 0, 1, 2$, the number of players that get payoff B_k in some stopping order (i.e. number of players who have k neighbors at the date when they stop). Every player eventually stops in each stopping order, so we must have $n_0 + n_1 + n_2 = n$.

Our model allows for the possibility that two neighbors stop simultaneously. However, that is never optimal in terms of total welfare, since by stopping sequentially (even with a negligible lag between the stopping decisions), the payoff of one of the two players jumps up from B_k to B_{k-1}. Therefore, when considering the stopping order that maximizes the total welfare, we ignore the possibility of simultaneous stopping, and take as the set of feasible stopping orders the set of permutations of the players in the segment.

We aim to express the total welfare from an arbitrary stopping order (permutation) as a function of n_2. As a first step, we consider all the feasible values of n_2 across all possible stopping orders. We claim that

$$\begin{aligned} n_2 &\in \left\{0, ..., \frac{N-1}{2}\right\} \text{ if } n \text{ is odd,} \\ n_2 &\in \left\{0, ..., \frac{N-2}{2}\right\} \text{ if } n \text{ is even.} \end{aligned} \tag{12}$$

To prove this claim, it suffices to note that the smallest possible value $n_2 = 0$ is trivially obtained in the case of a shrinking network, i.e. in a case where the players stop in the order 1, 2, 3, ... (or alternatively n, $n-1$, $n-2$, ...). The largest possible value for n_2 is obtained in any sequence where all the even numbered players stop before the odd numbered players (i.e. a regular fragmenting). Such a sequence gives $n_2 = \frac{n-1}{2}$ if n is odd and $n_2 = \frac{n-2}{2}$ if n is even. Any interior value for n_2 is obtained, for example, in a sequence, where the first n_2 even numbered players stop first (each of those players gets B_2), and after that all the remaining players stop in an increasing sequence (and get either B_0 or B_1 each).

As a second step, we claim that in any stopping order, we have

$$n_0 = n_2 + 1. \tag{13}$$

We prove this claim by induction. For $n = 1$ and $n = 2$, the only feasible values of n_0 and n_2 are obviously $n_0 = 1$ and $n_2 = 0$, so (13) holds. Take a line segment of length n where $n = 3, 4, ...$, and suppose as an induction hypothesis that (13) holds for all line segments of length $k < n$. There are two cases to consider. First, if the first player to stop in that segment is either player 1 or n (i.e. one of the end nodes), then this player gets B_1 and the line shrinks to length $n - 1$. By the induction hypothesis, (13) holds for that reduced line and therefore (13) holds for the original line of length n as well. Second, if the first player to stop is amongst players $2, ..., n-1$, she gets B_2 and the line segment splits into two shorter line segments of lengths n' and n'' with $n' + n'' = n - 1$. By the induction hypothesis, we have $n'_0 = n'_2 + 1$ and $n''_0 = n''_2 + 1$, where n'_k and n''_k denote the number of players in the two shorter line segments that get payoff B_k. The total number of players that get B_2 is then $n_2 = 1 + n'_2 + n''_2$ (where 1 is added because the first player to stop in the original line did get B_2) and the total number of players that get B_0 is $n_0 = n'_0 + n''_0$. Combining these equations gives $n_0 = n_2 + 1$ i.e. (13) holds for the original line segment as well.

As a third step, we claim that in any stopping order, we have

$$n_1 = n - 2n_2 - 1. \tag{14}$$

This follows simply from combining (13) and $n = n_0 + n_1 + n_2$, and solving for n_1.

We can now use (13) and (14) to compute the total welfare in an arbitrary stopping order as a function of n_2:

$$\begin{aligned} W(n_2) &= n_0 B_0 + n_1 B_1 + n_2 B_2 \\ &= (n_2 + 1) B_0 + (N - 2n_2 - 1) B_1 + n_2 B_2 \\ &= B_0 + (N - 1) B_1 + (B_0 + B_2 - 2B_1) n_2. \end{aligned}$$

We can see from this equation that if $2B_1 < B_0 + B_2$, the total welfare is maximized by choosing the highest possible value of n_2. By (12) this is obtained with regular fragmenting, where every even numbered player stops first. If $2B_1 > B_0 + B_2$, then the total welfare is maximized by choosing $n_2 = 0$, which is obtained in a shrinking network.

Proof of Proposition 5

We start by deriving the fraction of surviving firms p_e. We already outlined a proof based on a difference equation approach in the main text. Here we present an alternative direct proof based on the main method of analysis of this paper.

As a first step we will determine the probability with which an arbitrary firm i exits before one of her neighbors does, a probability that we denote by ω. For $t < t_2$ firm i exits with a hazard rate $\lambda_2(t)$ as long as none of her two neighbors have exited. Denote by $f(t)$ the probability density function for i's planned exit time (i.e. time to exit if none of her neighbors have yet stopped):

$$f(t) = \lambda_2(t)\, e^{-\int_0^t \lambda_2(s)ds}.$$

Given the information i has, the perceived hazard rate with which a neighbor of i exits is $\gamma(s)$ so that the probability that none of i's neighbors have exited at time t (given that i has not) is

$$e^{-\int_0^t 2\gamma(s)ds}.$$

Using this, we can write the probability that i exits before one of her neighbors as:

$$\omega = \int_0^{t_2} f(t)\cdot e^{-\int_0^t 2\gamma(s)ds}dt = \int_0^{t_2} \lambda_2(t)\cdot e^{-\int_0^t \lambda_2(s)ds}\cdot e^{-\int_0^t 2\gamma(s)ds}dt. \tag{15}$$

To evaluate this expression, we utilize the connection between $\lambda_2(s)$ and $\gamma(s)$. We know that:

$$\lambda_2(t) = \frac{\gamma(t)}{p_2(t)}, \tag{16}$$

where $p_2(t)$ evolves according to

$$\dot{p}_2(t) = -\lambda_2(t)\cdot p_2(t),$$

which can be solved with boundary condition $p_2(0) = 1$ to get a closed form formula for $p_2(t)$:

$$p_2(t) = e^{-\int_0^t \lambda_2(s)ds}. \tag{17}$$

From (16) and (17) we then have

$$\lambda_2(t)\cdot e^{-\int_0^t \lambda_2(s)ds} = \gamma(t),$$

so that (15) reduces to

$$\omega = \int_0^{t_2} \gamma(t)\cdot e^{-\int_0^t 2\gamma(s)ds}dt. \tag{18}$$

Noting that

$$\begin{aligned}\frac{d}{dt} - \int_0^t 2\gamma(s)\,ds &= -2\gamma(t)\,, \text{ and} \\ \int_0^{t_2} \gamma(s)\,ds &= 1,\end{aligned}$$

we can solve (18) to get

$$\omega = \frac{1}{2}\left(1 - e^{-2}\right) \approx 0.43.$$

Finally, using the same reasoning as in the proof of Proposition 4, we note that in an infinite line, the fraction of players that get payoff B_2 is the same as the fraction of players that get payoff B_0. Therefore, we have $p_e = \omega$, which establishes the first result.

We now determine the distribution of random variable l_g. As a first step, consider the gaps at time t_2, i.e. the end of the first phase of the game. At that point, every gap is of length 1 and each sequence of two gaps is separated either by an individual remaining player or a pair of remaining players. Each gap may either remain a gap of length $l_g = 1$ or develop into a gap of length $l_g = 2$ or $l_g = 3$ as $t \to \infty$.

Let us take an arbitrary gap at time t_2. Consider the probability that to the right of this gap there is a pair of players instead of one isolated player, and denote this probability by p. Noting that the fraction of gaps to the remaining players is ω, we can compute p by noting that the expected number of players (including the gap) until the next gap is $1 + (1-p) \cdot 1 + p \cdot 2 = 2 + p$. Therefore, the fraction of gaps ω to all the nodes can be expressed as

$$\omega = \frac{1}{2+p}.$$

Combining this with our earlier expression $\omega = \frac{1}{2}\left(1 - e^{-2}\right)$ and solving for p gives

$$p = 2\frac{1}{1+e^2}.$$

By symmetry and independence of types of neighbors, p is also the probability that there is a pair of remaining players to the left of the gap. With this information, we are in a position to derive the probability distribution of l_g. Consider the length of the gap at $t \to \infty$. A gap $l_g = 3$ can only occur at the end of the game if to the right and to the left of the initial gap (probability p^2), there was a pair, and the firms closer to the gap exited (probability $\frac{1}{4}$). For a gap of size two to appear, you need at least one pair. The distribution is thus given as in the main text:

$$\begin{aligned} P\left[l_g=3\right] &= p^2\frac{1}{4}, \\ P\left[l_g=2\right] &= p^2\frac{1}{2}+2p(1-p)\frac{1}{2}, \\ P\left[l_g=1\right] &= p^2\frac{1}{4}+2p(1-p)\frac{1}{2}+(1-p)^2. \end{aligned}$$

Proof of Proposition 6

We first compute the expected subsidy payment separately for each type. For type 0, the payment is made immediately, so the cost is simply s. For type $k=1$, payment accrues at time τ that is exponential with parameter $\lambda_1(s)+\gamma^{pe}(s)$, so the discounted cost is

$$\mathbb{E}\left(e^{-r\tau}s\right)=s\int_0^\infty\left(\lambda_1^{pe}(s)+\gamma^{pe}(s)\right)e^{-(\lambda_1(s)+\gamma^{pe}(s))t}e^{-rt}dt=\frac{\lambda_1^{pe}(s)+\gamma^{pe}(s)}{\lambda_1^{pe}(s)+\gamma^{pe}(s)+r}s.$$

Type $k=2$ becomes type $k=1$ at time τ_1 that is exponential with parameter $2\gamma^{pe}(s)$, and then will wait another time interval τ to stop. The expected payment is therefore

$$\mathbb{E}\left(e^{-r(\tau_1+\tau)}s\right)=\mathbb{E}\left(e^{-r\tau_1}\right)\mathbb{E}\left(e^{-r\tau}\right)s=\frac{2\gamma^{pe}(s)\left(\lambda_1^{pe}(s)+\gamma^{pe}(s)\right)}{\left(2\gamma^{pe}(s)+r\right)\left(\lambda_1^{pe}(s)+\gamma^{pe}(s)+r\right)}s.$$

Weighting the cost terms with the population shares of different types, the expected discounted subsidy payment to an arbitrary player is:

$$C^{pe}(s)=q_0s+q_1\frac{\lambda_1^{pe}(s)+\gamma^{pe}(s)}{\lambda_1^{pe}(s)+\gamma^{pe}(s)+r}s+q_2\frac{2\gamma^{pe}(s)\left(\lambda_1^{pe}(s)+\gamma^{pe}(s)\right)}{\left(2\gamma^{pe}(s)+r\right)\left(\lambda_1^{pe}(s)+\gamma^{pe}(s)+r\right)}s.$$

Substituting in $\gamma^{pe}(s)$ and $\lambda_1^{pe}(s)$ from (8), using $q_0+q_1+q_2=1$, and simplifying, we can write this as:

$$C^{pe}(s)=s\frac{q_0\left(B_0-B_1\right)+2\left(B_1+s\right)}{B_0+B_1+2s}.$$

Plugging this and (9) in (7), the total welfare is:

$$\begin{aligned} W^{pe}(s) &= q_0\left(B_0+s\right)+q_1\left(B_1+s\right)+q_2\frac{2\left(B_1+s\right)^2}{B_0+B_1+2s} \\ &\quad -(1+\alpha)\,s\frac{q_0\left(B_0-B_1\right)+2\left(B_1+s\right)}{B_0+B_1+2s}. \end{aligned}$$

It is easy to show by direct computation that $W^{pe}(s)$ is concave in s, and

$$\begin{aligned} \left(W^{pe}\right)'(0) &\quad > \quad (<)\,0 \\ &\Longleftrightarrow \\ \alpha &\quad < \quad (>)\,a^{*}, \end{aligned}$$

where

$$\alpha^{*} := \frac{q_0 2B_1^2 + q_1\left(B_0^2 + B_1^2\right) + q_2 2B_0 B_1 - 2B_1^2}{\left(B_0 + B_1\right)\left(\left(B_0 - B_1\right) q_0 + 2B_1\right)} > 0.$$

Since $W^{pe}(0)$ gives the total welfare without any subsidy, this proves Proposition 6.

Proof of Proposition 7

We write the total welfare as

$$W^{ra}(s,\kappa) = G^{ra}(s,\kappa) - (1+\alpha)\,C^{ra}(s,\kappa),$$

where $G^{ra}(s,\kappa)$ is the welfare of the players and $C^{ra}(s,\kappa)$ is the financial cost of the policy.

Consider first the welfare term $G^{ra}(s,\kappa)$. As before, we can write it as

$$G^{ra}(s,\kappa) = q_0\left(B_0 + s\right) + q_1\left(B_1 + s\right) + q_2 V^{ra}(s,\kappa),$$

where $V^{ra}(s,\kappa)$ is the value of type $k=2$. Since both of her neighbors stop with hazard rate $\gamma^{ra}(s,\kappa)$ until the subsidy expires, she observes an exit at rate $2\gamma^{ra}(s,\kappa)$. If one of her neighbors exit before expiration, she gets $B_1 + s$, otherwise she will get value

$$V_2 = \frac{2B_1^2}{B_0 + B_1}$$

at the time of expiration of the policy. Therefore, we get

$$\begin{aligned} V^{ra}(s,\kappa) &= \int_0^\infty \kappa e^{-\kappa t}\left[\int_0^t 2\gamma^{ra}(s,\kappa)\,e^{-2\gamma^{ra}(s,\kappa)u} e^{-ru}\left(B_1 + s\right) du + e^{-rt} e^{-2\gamma^{ra}(s,\kappa)t} V_2\right] dt \\ &= \frac{2\gamma^{ra}(s,\kappa)\left(B_1 + s\right) + \kappa V_2}{2\gamma^{ra}(s,\kappa) + \kappa + r} = \frac{2\gamma^{ra}(s,\kappa)\left(B_1 + s\right) + \kappa\frac{2B_1^2}{B_0 + B_1}}{2\gamma^{ra}(s,\kappa) + \kappa + r}. \end{aligned}$$

We write the cost term $C^{ra}(s,\kappa)$ as

$$C^{ra}(s,\kappa) = q_0 s + q_1 C_1^{ra}(s,\kappa) + q_2 C_2^{ra}(s,\kappa),$$

and proceed to compute the terms $C_1^{ra}(s,\kappa)$ and $C_2^{ra}(s,\kappa)$ corresponding to types $k=1$ and $k=2$. For a player i of type $k=1$, the subsidy has to be paid either if i) i exits before subsidy expires, or ii) i's only neighbor exits before subsidy expires. Event i) arrives at rate

$$\lambda_1^{ra}(s,\kappa)=\frac{\gamma_1^{ra}(s,\kappa)}{p_1(0)}=\frac{\gamma_1^{ra}(s,\kappa)(q_1+2q_2)}{q_1},$$

and event ii) arrives at rate $\gamma_1^{ra}(s,\kappa)$. We have

$$\gamma_1^{ra}(s,\kappa)+\lambda_1^{ra}(s,\kappa)=\gamma_1^{ra}(s,\kappa)\left(1+\frac{q_1+2q_2}{q_1}\right)=2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1},$$

and so we can write $C_1^{ra}(s,\kappa)$ as

$$\begin{aligned}C_1^{ra}(s,\kappa) &= \int_0^\infty \kappa e^{-\kappa t}\left[\int_0^t 2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}e^{-2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}u}e^{-ru}s\,du\right]dt\\ &= s\frac{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}}{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}+\kappa+r}.\end{aligned}$$

To compute $C_2^{ra}(s,\kappa)$, note that for type $k=2$ to obtain the subsidy, one of her two neighbors must first exit before subsidy expires. This event arrives at rate $2\gamma_1^{ra}(s,\kappa)$, and once it happens the player turns into type $k=1$ and the continuation subsidy cost at that date is $C_1^{ra}(s)$. Hence, we have

$$\begin{aligned}C_2^{ra}(s,\kappa) &= \int_0^\infty \kappa e^{-\kappa t}\left[\int_0^t 2\gamma_1^{ra}(s,\kappa)e^{-2\gamma_1^{ra}(s,\kappa)u}e^{-ru}C_1^{ra}(s)\,du\right]dt\\ &= \frac{2\gamma_1^{ra}(s,\kappa)C_1^{ra}(s)}{2\gamma_1^{ra}(s,\kappa)+\kappa+r}=s\frac{2\gamma_1^{ra}(s,\kappa)\frac{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}}{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}+\kappa+r}}{2\gamma_1^{ra}(s,\kappa)+\kappa+r}.\end{aligned}$$

The total welfare is then:

$$\begin{aligned}W^{ra}(s,\kappa) &= q_0(B_0+s)+q_1(B_1+s)+q_2\left[\frac{2\gamma^{ra}(s,\kappa)(B_1+s)+\kappa\frac{2B_1^2}{B_0+B_1}}{2\gamma^{ra}(s,\kappa)+\kappa+r}\right]\\ &\quad-(1+\alpha)s\left[q_0+q_1\frac{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}}{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}+\kappa+r}+q_2\frac{2\gamma_1^{ra}(s,\kappa)\frac{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}}{2\gamma_1^{ra}(s,\kappa)\frac{q_1+q_2}{q_1}+\kappa+r}}{2\gamma_1^{ra}(s,\kappa)+\kappa+r}\right].\end{aligned}$$

Plugging in

$$\gamma^{ra}(s,\kappa) = \frac{r(B_1+s)+\kappa s}{B_0-B_1}$$

it is straightforward to show by direct computation that

$$\frac{\partial W^{ra}(s,\kappa)}{\partial \kappa} > 0,$$

and

$$\lim_{\kappa\to 0}\left(W^{ra}(s;\kappa)\right) = W^{pe}(s).$$

We can now also compute the total welfare in the limit $\kappa \to \infty$ as:

$$\begin{aligned} W^{ra}_{\infty}(s) \quad : \quad & = \lim_{\kappa\to\infty} W^{ra}(s;\kappa) = q_0(B_0+s) + q_1(B_1+s) \\ & + q_2\left[\frac{2s(B_1+s)}{B_0-B_1+2s} + \frac{B_0-B_1}{B_0-B_1+2s}\frac{2B_1^2}{B_0+B_1}\right] \\ & -(1+\alpha)s\cdot\left[q_0 + q_1\left(\frac{2s(q_1+q_2)}{2s(q_1+q_2)+q_1(B_0-B_1)}\right)\right. \\ & \left. + q_2\left(\frac{2s}{2s+B_0-B_1}\frac{2s(q_1+q_2)}{2s(q_1+q_2)+q_1(B_0-B_1)}\right)\right]. \end{aligned}$$

Proof of Proposition 8

Consider a subsidy policy that starts from level s and declines linearly to zero:

$$s(t) = \begin{cases} s-\xi t \text{ for } t \leq \frac{s}{\xi} \\ 0 \text{ for } t > \frac{s}{\xi} \end{cases},$$

where ξ is a parameter that defines the speed of decline. The hazard rate of exit is then

$$\gamma^{sm}(t) = \frac{r(B_1+s(t)) - \dot{s}(t)}{B_0-B_1} = \frac{r(B_1+s(t))}{B_0-B_1} + \frac{\xi}{B_0-B_1},$$

which is linear in ξ. We will analyze the limit $\xi \to \infty$. Note that in this limit the second term of the hazard rate explodes while the first term is bounded and becomes negligible relative to the second term. Therefore, we approximate the effect of the policy by ignoring the first term and replacing the actual hazard rate of exit by

$$\overline{\gamma}(\xi) := \frac{\xi}{B_0-B_1}.$$

Since the policy is in effect for a vanishingly short time period $\left[0, \frac{s_0}{\xi}\right]$, the error due to this approximation vanishes in the limit $\xi \to \infty$. Let

$$\overline{\lambda}(\xi) := \frac{\overline{\gamma}(\xi)}{p_1(0)} = \frac{q_1 + 2q_2}{q_1}\overline{\gamma}(\xi)$$

denote the corresponding approximate stopping rate of an individual player of type $k = 1$.

Consider now the benefits of such a policy. Type 0 will stop immediately and type 1 is indifferent between stopping immediately, so the payoffs to those types are given by $B_0 + s$ and $B_1 + s$, as before. Denote payoff of type 2 by $V^{sm}(s)$. Since both of her neighbors stop with hazard rate $\overline{\gamma}(\xi)$ within $\left[0, \frac{s}{\xi}\right]$, she observes an exit at rate

$$2\overline{\gamma}(\xi) = \frac{2\xi}{B_0 - B_1}$$

within that interval, and gets value

$$B_1 + s(t) = B_1 + s - \xi t$$

at the date when a neighbor exits. Since the length of the subsidy period $\left[0, \frac{s_0}{\xi}\right]$ is negligible when ξ is large, we can ignore discounting within the period, and compute

$$\begin{aligned} V_2^{sm}(s) &= \int_0^{\frac{s}{\xi}} (B_1 + s - \xi t) \frac{2\xi}{B_0 - B_1} e^{-\frac{2\xi}{B_0 - B_1}t} dt \\ &\quad + \left(1 - \int_0^{\frac{s}{\xi}} \frac{2\xi}{B_0 - B_1} e^{-\frac{2\xi}{B_0 - B_1}t} dt\right) V_2 + \Delta(\xi) \\ &= \left(\frac{3}{2}B_1 - \frac{1}{2}B_0\right)\left(1 - e^{-\frac{2s}{B_0 - B_1}}\right) + s_0 + e^{-\frac{2s}{B_0 - B_1}} \frac{2B_1^2}{B_0 + B_1} + \Delta(\xi), \end{aligned}$$

where $\Delta(\xi) \to 0$ as $\xi \to \infty$. Denoting by $G_\infty^{sm}(s)$ the welfare of the players in the limit $\xi \to \infty$, we have then

$$\begin{aligned} G_\infty^{sm}(s) &= q_0(B_0 + s) + q_1(B_1 + s) \\ &\quad + q_2\left[\left(\frac{3}{2}B_1 - \frac{1}{2}B_0\right)\left(1 - e^{-\frac{2s}{B_0 - B_1}}\right) + s + e^{-\frac{2s}{B_0 - B_1}} \frac{2B_1^2}{B_0 + B_1}\right]. \end{aligned}$$

Consider next the financial costs of the policy. For type 0 cost is s as with other policies. For player i who is type 1, subsidy has to be paid either if i) i exits before subsidy expires, or ii) i's only neighbor exits before subsidy expires. Event i) arrives at rate

$$\overline{\lambda}(\xi) = \frac{\overline{\gamma}(\xi)(q_1 + 2q_2)}{q_1}$$

and event ii) arrives at rate γ^{∞}. We have

$$\begin{aligned}
\overline{\gamma}(\xi) + \overline{\lambda}(\xi) &= \overline{\gamma}(\xi)\left(1 + \frac{q_1 + 2q_2}{q_1}\right) = 2\overline{\gamma}(\xi)\frac{q_1 + q_2}{q_1} \\
&= \frac{2\xi(q_1 + q_2)}{q_1(B_0 - B_1)},
\end{aligned}$$

and so the expected subsidy payment conditional on it occurring is

$$\begin{aligned}
C_1^{sm}(s) &= \int_0^{\frac{s}{\xi}} (s - \xi t)\frac{2\xi(q_1 + q_2)}{q_1(B_0 - B_1)} e^{-\frac{2\xi(q_1+q_2)}{q_1(B_0-B_1)}t} dt + \Delta'(\xi) \\
&= \frac{(B_0 - B_1) q_1 \left(e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}} - 1\right)}{2(q_1 + q_2)} + s + \Delta'(\xi),
\end{aligned}$$

where again $\Delta'(\xi) \to 0$ as $\xi \to \infty$. Type 2 will obtain subsidy if i) one of her two neighbors exits before subsidy expires, and ii) she herself exits before subsidy expires. The first event arrives at rate

$$2\overline{\gamma}(\xi) = \frac{2\xi}{B_0 - B_1},$$

and so, noting that $C_1^{sm}(s - \xi t)$ is the expected payment to a player that becomes type 1 at time $t \leq \frac{s}{\xi}$ and using iterated law of expectation, we can write the expected payment to type 2 as

$$\begin{aligned}
C_2^{sm}(s) &= \int_0^{\frac{s}{\xi}} C_1^{sm}(s - \xi t)\frac{2\xi}{B_0 - B_1} e^{-\frac{2\xi}{B_0-B_1}t} dt + \Delta''(\xi) \\
&= \int_0^{\frac{s}{\xi}} \left(\frac{(B_0 - B_1) q_1 \left(e^{-\frac{2(s-\xi t)(q_1+q_2)}{q_1(B_0-B_1)}} - 1\right)}{2(q_1 + q_2)} + (s - \xi t)\right)\frac{2\xi}{B_0 - B_1} e^{-\frac{2\xi}{B_0-B_1}t} dt + \Delta''(\xi) \\
&= \frac{(B_0 - B_1)\left(\frac{(q_1+q_2)^2 e^{-\frac{2s}{B_0-B_1}} - (q_1)^2 e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}}}{q_2} - (2q_1 + q_2)\right)}{2(q_1 + q_2)} + s + \Delta''(\xi),
\end{aligned}$$

where again $\Delta''(\xi) \to 0$ as $\xi \to \infty$. The expected total cost in the limit $\xi \to \infty$ is then

$$\begin{aligned} C_{\infty}^{sm}(s) &= q_0 s + q_1 \left[\frac{(B_0 - B_1) q_1 \left(e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}} - 1 \right)}{2(q_1+q_2)} + s \right] \\ &\quad + q_2 \left[\frac{(B_0 - B_1) \left(\frac{(q_1+q_2)^2 e^{-\frac{2s}{B_0-B_1}} - (q_1)^2 e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}}}{q_2} - (2q_1 + q_2) \right)}{2(q_1+q_2)} + s \right] \end{aligned}$$

and the total welfare is

$$\begin{aligned} W_{\infty}^{sm}(s) &= q_0 (B_0 + s) + q_1 (B_1 + s) \\ &\quad + q_2 \left[\left(\frac{3}{2} B_1 - \frac{1}{2} B_0 \right) \left(1 - e^{-\frac{2s}{B_0-B_1}} \right) + s + e^{-\frac{2s}{B_0-B_1}} \frac{2B_1^2}{B_0+B_1} \right] \\ &\quad - (1+\alpha) s \left\{ q_0 s + q_1 \left[(B_0 - B_1) q_1 \left(e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}} - 1 \right) + s \right] \right. \\ &\quad \left. + q_2 \left(1 - e^{-\frac{2s}{B_0-B_1}} \right) \left[\frac{(B_0 - B_1) q_1 \left(e^{-\frac{2s(q_1+q_2)}{q_1(B_0-B_1)}} - 1 \right)}{2(q_1+q_2)} + s \right] \right\}. \end{aligned}$$

By direct computation, we find that W^{sm} is strictly concave:

$$\left(W_{\infty}^{sm} \right)''(s) < 0$$

for all $s \geq 0$. Moreover, we find

$$\begin{aligned} & W_{\infty}^{sm}(s) - W_{\infty}^{ra}(s) \\ = & \frac{(B_0 - B_1) e^{-\frac{2s}{B_0-B_1}} \left[(B_0 - B_1) \left(e^{\frac{2s}{B_0-B_1}} - 1 \right) - 2s \right] \left[(B_0 + B_1)(q_1(1+\alpha) + q_2\alpha) + 2q_2\alpha \right]}{2(B_0+B_1)(B_0 - B_1 + 2s)} > 0. \end{aligned}$$

Proof of Proposition 9

We show that any constant adoption rate γ implemented by a permanent subsidy s handed to any adopting agent is also implemented by a restricted permanent subsidy s^* handed only to adopting type 1 agents, with s^*, with $s^* < s$. This will then establish that the social planner can implement any given constant neighbor adoption rate γ at lower cost under a neighbor reward policy than under a permanent subsidy.

For any $\gamma \geq \overline{\gamma}_1$, the permanent subsidy s required to implement γ is

$$s = (B_0 - B_1)\left(\frac{\gamma}{r}\right) - B_1 \geq 0$$

whereas the restricted subsidy s^* required to implement the same neighbor adoption rate γ satisfies

$$s^* = \frac{(B_0 - B_1)\left(\frac{\gamma}{r}\right) - B_1}{\frac{\gamma}{r} + 1} \in [0, s)\,.$$

These two policies implement an identical joint distribution of adoption rates and type at the time of adoption. The amount s^* of the restricted permanent subsidy is lower than the amount s of the permanent subsidy. Moreover, the restricted permanent subsidy is paid only to agents who are of type 1 at the time of adopting, unlike the permanent subsidy, which is also paid to type 0. It follows that the restricted permanent subsidy s^* yields a higher social welfare than the permanent subsidy s. Moreover, the restricted subsidy s^* is payoff equivalent to the neighbor reward program m, with

$$m = \left(1 + \frac{p_1}{\frac{\gamma}{r}}\right) s^*.$$

In other words, the permanent subsidy $s > 0$ yields a lower social welfare than the neighbor reward

$$m = \left(1 + \frac{p_1\,(B_0 - B_1)}{s + B_1}\right) \frac{s\,(B_0 - B_1)}{s + B_0},$$

which is an amount in $[0, B_0 - B_1]\,.$

8 Appendix B: NOT FOR PUBLICATION

B1: Informational spillovers

We present a specific model involving informational spillovers across neighbors in the case of the line. Suppose that the cost of adopting depends on the technique used. There are two choices to be made when adopting, for instance different organizational dimensions, $a_1 \in \{L, R\}$ and $a_2 \in \{L, R\}$. The state of nature, described by $\theta = \{\theta_1, \theta_2\}$ determines which adoption technique is less costly. Specifically, the cost of adoption is $c = c_1 + c_2$ where $c_i = c_l 1_{a_i = \theta_i} + c_h 1_{a_i \neq \theta_i}$, i.e the cost is minimized when the technique used matches the state. When a player observes her neighbor, with probability 1/2, she learns perfectly about dimension 1 and with probability 1/2 about dimension 2. What is learned does not depend on the choice the neighbor actually made, which ensures that there is no inference made on the information the neighbor's neighbor held.

In this case

$$
\begin{aligned}
B_2 &= B - 2\frac{1}{2}(c_l + c_h) = B - (c_l + c_h), \\
B_1 &= B - c_l - \frac{1}{2}(c_l + c_h) = B - (\frac{3}{2}c_l + \frac{1}{2}c_h), \\
B_0 &= B - c_l - \frac{1}{2}(c_l) - \frac{1}{4}(c_l + c_h) = B - (\frac{7}{4}c_l + \frac{1}{4}c_h).
\end{aligned}
$$

So that

$$
\begin{aligned}
B_0 - B_1 &= \frac{1}{4}(c_h - c_l), \\
B_1 - B_2 &= \frac{1}{2}(c_h - c_l).
\end{aligned}
$$

In this case we have $\gamma_1 > \gamma_2$, so that this setup will naturally correspond to the shrinking network setup.

B2: War of attrition

We present here a more classical version of the war of attrition, adding as in the rest of the paper the network structure. Firms decide when to exit, where exit is irreversible. Staying in costs $c > 0$ per unit of time, but there is no discounting.

Once both neighbors of a firm exit, the remaining isolated firm gets prize B. As in the rest of the paper, each player only observes whether her neighbors are active or not, but cannot see the status of any other player in the network.

We show there exists a symmetric equilibrium, characterized by a date $t' > 0$ such that within $(0, t')$ all those players who have two active neighbors mix, and within (t', ∞)

there are only players with one active neighbor left (i.e. isolated pairs of players) who play a standard war of attrition with each other.

Denote by $V(t)$ the value of a player, who has one active neighbor left (so that one of her two neighbors have exited). We have $V(t) > 0$ for $t \in (0, t')$ and $V(t') = 0$.

Let us denote by $\gamma(t)$ the hazard rate with which an arbitrary neighbor exits at time t, where $t \in (0, t')$. For a randomizing player to be indifferent, the benefit of delaying exit by dt must equate the cost of doing so, i.e. $2\gamma(t)\,dtV(t) = cdt$, so that

$$2\gamma(t)\,V(t) = c,$$

or

$$\gamma(t) = \frac{c}{2V(t)}. \tag{19}$$

The Bellman equation for the player who has only one neighbor left can be written:

$$V(t) = \gamma(t)\,dtB + (1 - \gamma(t)\,dt)\left(V(t) + \dot{V}(t)\,dt\right) - cdt,$$

which gives

$$\dot{V}(t) = -\gamma(t)\,(B - V(t)) + c. \tag{20}$$

Plugging (19) in (20) gives us a differential equation for $V(t)$:

$$\dot{V}(t) = -\frac{cB}{2V(t)} + \frac{3}{2}c.$$

Starting with any initial value $V(0)$ such that $0 < V(0) < \frac{B}{3}$ this has a solution $V(t)$ that is decreasing and hits zero at some time point t'.

B3: Generalization with two state variables

In the application to the adoption of technologies, a more general model should keep track of two state variables:

- a the number of active neighbors
- i the number of inactive neighbors

Types are thus described by (a, i) where $a \in \{0, 1, 2\}$ and $i \in \{0, 1, 2\}$. A random member of the network can be of types $(2, 0)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, $(0, 1)$ or $(0, 0)$. In the model used in the core of the paper, we restrict ourselves to one state variable. The implicit assumption we make is that $a + i = 2$, i.e everyone starts with the same number of neighbors, some active and some inactive. Thus in the main part of the paper there were only three possible types $(2,0)$, $(1,1)$, and $(0,2)$. We now show that the general pattern is preserved with a slight complication due to the existence of types $(1,0)$. Types $(0,2)$, $(0,1)$ or $(0,0)$ do not have any active neighbors and therefore stop immediately regardless whether they have 0, 1 or 2 inactive neighbors.

As in the main model we introduce some important measures:

$$\overline{\gamma}_{(1,0)} := \frac{rB_2}{B_1 - B_2},$$
$$\overline{\gamma}_{(1,1)} := \frac{rB_1}{B_0 - B_1},$$
$$\overline{\gamma}_{(2,0)} := \frac{rB_2}{2(B_1 - B_2)}.$$

The equivalence between types here and in the model of section 3 implies that $\overline{\gamma}_{(1,1)} = \overline{\gamma}_1$ and $\overline{\gamma}_{(2,0)} = \overline{\gamma}_2$. We consider two cases: $\overline{\gamma}_{(1,0)} > \overline{\gamma}_{(1,1)}$ and $\overline{\gamma}_{(1,0)} < \overline{\gamma}_{(1,1)}$.

Case 1: $\overline{\gamma}_{(1,0)} > \overline{\gamma}_{(1,1)}$

In this case types $(1,0)$ have the highest incentives to stop. Indeed these types always have a higher incentive to stop than types $(2,0)$, since they get the same benefit from stopping B_2, but they get lower benefit of waiting $\mu(B_1 - B_2)$, whereas types $(2,0)$ get benefit $(2\mu(V_1 - B_2)$ with $V_1 > B_1)$. We now describe the evolution of beliefs.

$$\dot{p}_{(1,0)}(t) = -\lambda(t)\, p_{(1,0)}(t) \left(1 - p_{(1,0)}(t)\right) < 0.$$

As time passes, players become less confident that their neighbor is of type $(1,0)$. Whereas in section 3 there were two countervailing forces affecting beliefs, here the second force is not present since types $(2,0)$, if their other neighbor happens to stop, will turn into a type $(1,1)$, not a type $(1,0)$.

Thus at some date $t_{(1,0)}$ all types $(1,0)$ will have stopped. We are then back to the case studied in section 3 with only types $(1,1)$ and $(2,0)$. Depending on the relative size of $\overline{\gamma}_{(1,1)} := \frac{rB_1}{B_0 - B_1}$ and $\overline{\gamma}_{(2,0)} := \frac{rB_2}{2(B_1 - B_2)}$, we will be either in the case of shrinking or of fragmenting networks.

Case 2: $\overline{\gamma}_{(1,1)} > \overline{\gamma}_{(1,0)}$

Types $(1,1)$ initially mix. The evolution of beliefs is given by:

$$\begin{aligned}\dot{p}_{(1,1)}(t) &= -\lambda(t)\, p_{(1,1)}(t)\left(1 - p_{(1,1)}(t)\right) + \overline{\gamma}_{(1,1)}(t)\, p_{(2,0)}(t) \\ &= -\overline{\gamma}_{(1,1)}\left(1 - p_{(1,1)}(t) - p_{(2,0)}(t)\right) < 0\end{aligned}$$

In this case, as in the case studied in section 3, there are two forces affecting the belief $p_{(1,1)}(t)$. However, the dominating effect is the evolution of beliefs and as time passes, active members of the network become less confident that their neighbor is of type $(1,1)$. At some date $t_{(1,1)}$, among active members of the networks, only types $(1,0)$ and $(2,0)$ remain. The networks are therefore formed of lines of random sizes with types $(1,0)$ at the extremities. Types $(1,0)$ then have a strictly higher incentive to adopt. As soon as a type $(1,0)$ adopts, the neighbor, if he is of type $(2,0)$, transforms into a type $(1,1)$ and thus immediately adopts. Thus entry by a type $(1,0)$ creates an immediate cascade that immediately covers the entire line. It is therefore as if types $(1,0)$ were playing a waiting game with no type uncertainty. They therefore mix at rate $\overline{\gamma}_{(1,0)}$ and as soon as one adopts, so does the entire line.

Appendix C: Non-Markovian equilibria

Both in the shrinking and fragmenting network cases, there can be non-Markovian Equilibria, where the agents use the realizations of their neighbor's exit dates as randomization devices for their own dates of exit. Importantly, such equilibria are associated with the same distribution $F(t)$ of dates at which a neighbor stops, and also with the same hazard rate $\gamma(t)$ at which a neighbor stops.

Shrinking networks

In the shrinking network case, a simple (and extreme) example of such an equilibrium is the following. In every component, the two players who start off as types 1 from the beginning of the game mix at constant rate $\lambda_1(t) = \overline{\gamma}_1$. Type 2 players never stop, unless one of their neighbors stops, in which case they follow immediately, as soon as they turn into types 1.

Under these strategies, the belief $p_1(t)$ about a neighbor remains constant equal to $p_1(0)$, exactly like in the case of this Markovian equilibrium, but for different reasons. First, a failure to stop is not informative about a neighbor's type, since a type 1 or a type 2 neighbor are equally likely to stop at any given time: a type 1 neighbor, on her own initiative, and a type 2 in reaction to her other neighbor's exit. Thus the belief updating effect is null. Second, a neighbor who was perviously a type 2 cannot possibly have turned into a type 1, otherwise she would have stopped immediately. Thus the evolving type effect is also absent. Overall, the belief about a neighbor's type remains constant.

In this non-Markovian equilibrium, the distribution of the stopping date of an average player is exponential with parameter $\frac{2\overline{\gamma}_1}{q_1+q_2 q_1+q_2}$, as in the Markovian case. As a result, the average time before an average member of the network stops is the same as in the Markovian equilibrium

$$E[T] = (q_1 + q_2) \frac{1}{2\overline{\gamma}_1}.$$

Note also that the distribution of the stopping date is the same for all agents initially of type either 1 or 2, since all agents of a given component stop at the same date.

This is different in the Markovian equilibrium. There, the distribution is different for players who are initially type 1 and the ones who are initially type 2. The distribution of the stopping date of an agent who is initially of type 1 is also exponential with rate $\frac{2(q_1+q_2)\overline{\gamma}_1}{q_1}$. For agents who are initially type 2, the stopping date is the sum of two variables, each of which follows an exponential distribution, the first with rate $2\overline{\gamma}_1$ and the second with rate $\frac{2(q_1+q_2)\overline{\gamma}_1}{q_1}$. But the distribution of the stopping date of an average player is the same in both the Markovian and non-Markovian equilibrium.

The main observable (and testable) difference between the two equilibria is in the joint distribution of the stopping time profiles. While the non-Markovian equilibrium example has all the agents exiting at the same date, there is some dispersion of exit dates in the Markovian equilibrium.[21]

Fragmenting networks

Fragmenting networks too admit non-Markovian equilibria, but their properties differ from the Markovian ones even less than in the shrinking case. In one instance of such an equilibrium, agents who are still active at date t_2 and are of type $k = 1$ at date t_2 could choose a stopping date that is an increasing function ϕ of the date at which their neighbor who stopped prior to t_2 did it. For an appropriately chosen function ϕ, this is an equilibrium.

The main properties of this equilibrium remain the same as for the Markovian equilibrium. One noticeable difference is the calculation of the distribution of the gap l_g from Proposition 5.

Indeed, when the equilibrium played is the one outlined in the previous paragraph, the events that two players separated by a gap at date t_2 and who are type 1 at that date both stop before their neighbors are no longer independent. As a result, the probability

[21] It is easy to construct other non-Markovian equilibria. For example, consider all the convex combinations of the Markovian equilibrium and our non-Markovian example. Or equilibria where players play the Markovian strategy in some set of dates and the non-Markovian example strategy in the complementary set of dates.

of this event is no longer $1/2 \cdot 1/2 = 1/4$ but it equals the probability that their common neighbor who stopped prior to date t_2 did it earlier than both of the neighbors of the neighbors of these two agents, which is a number p' greater than $1/3$.

Similarly, the events that two players separated by a gap at date t_2 and who are type 1 at that date both stop later than their neighbors are no longer independent. As a result, the probability of this event is no longer $1/2 \cdot 1/2 = 1/4$ but it equals the probability that their common neighbor who stopped prior to date t_2 did it earlier than both of the neighbors of the neighbors of these two agents, which is a number p'' smaller than $1/3$.

As a result, we obtain instead the following probabilities:

$$\begin{aligned} P\left[l_g = 3\right] &= p^2 p' > 0.02 \\ P\left[l_g = 2\right] &= p^2 \left(1 - p' - p''\right) + p\left(1 - p\right) \\ P\left[l_g = 1\right] &= p^2 p'' + p\left(1 - p\right) + \left(1 - p\right)^2 < 0.78. \end{aligned}$$