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A PROOF OF BLACKWELL'S THEOREM

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A Proof of Blackwell's Theorem^{*}

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Abstract

This note gives a new proof of Blackwell's celebrated result. The result is a bit stronger than the classical version since the action set and the prior are fixed, and only the utility of the decision maker varies. I show directly that a decision maker has access to a larger set of joint distributions over actions and states of the world if and only if her information improves in the garbling order.

1 Introduction

This note provides a proof of Blackwell's theorem (Blackwell, 1951, 1953). If a decision maker is identified with a prior on the states of the world, an action set, and a utility function over actions and states of the world, Blackwell's theorem says that an experiment π , that provides information about the state of the world, is preferred by every decision maker to an experiment π' if and only if π' is a garble of π . The proof I provide is relatively simple, and has the merit of making the intuitive point that the choice set of the decision maker is enlarged by moving from π' to π if and only if π' is a garble of π , which is absent from other proofs (Blackwell, 1951, 1953; Ponsard, 1975; Cremer, 1982; Leshno and Spector, 1992). Another advantage of this proof is that it varies only the utility of the decision maker, and not the prior or the action

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set¹, so the difficult direction of the result (π more useful than π' implies that π' is a garble of π) is slightly stronger here than in Blackwell's original formulation.

2 Setup and Preliminary Results

There is a finite action set A , with $|A| \geq 2$, and a finite set of states of the world Ω . The prior is a probability distribution $p(\omega)$ in $\Delta(\Omega)$. The payoff of the decision maker is given by a real valued payoff function $u(a, \omega)$. Let U be the set of such payoff functions. An experiment is given by a random variable x , with finite support X and a joint distribution function π on $X \times \Omega$ with marginal $p(\cdot)$ on Ω . When the decision maker can observe the realization of x , but not that of ω , she has access to mixed strategies $\sigma(a|x)$, with $\sum_a \sigma(a|x) = 1$ for all x . Let $\Sigma(\pi)$ be the set of strategies accessible to a decision maker endowed with experiment π . Ultimately, the decision maker only cares about the joint distributions of actions and states of the world, $\varphi(a, \omega)$. Let $\Phi(\pi)$ be the set of joint distributions she can generate when endowed with π , or *policy space*. It is restricted by her lack of knowledge in the following way:

$$\Phi(\pi) = \left\{ \varphi(a, \omega) : \exists \sigma \in \Sigma(\pi), \varphi(a, \omega) = \sum_x \sigma(a|x) \pi(x, \omega) \right\}.$$

It is easy to show that this set is a compact, and convex subset of $[0, 1]^{|A| \times |\Omega|}$. Then the problem of decision maker u endowed with experiment π is given by the following linear program

$$V(\pi, u) = \max_{\varphi \in \Phi(\pi)} \sum_{a, \omega} \varphi(a, \omega) u(a, \omega).$$

Definition 1 (Usefulness Order). *I say that an experiment π is more useful than another experiment π' , and write $\pi \succeq \pi'$, if all decision makers get a higher value when endowed with π than when they are endowed with π' , that is,*

$$\pi \succeq \pi' \Leftrightarrow V(\pi, u) \geq V(\pi', u), \forall u \in U$$

¹Leshno and Spector (1992) also fix the action set, but use a different proof technique based on matrices.

Definition 2 (Garbling Order). *I say that π' is a garble of π , and write $\pi' \trianglelefteq \pi$ if there exists a function $f : X \times X' \rightarrow [0, 1]$ such that $\pi'(x', \omega) = \sum_x f(x, x')\pi(x, \omega)$ and $\sum_{x'} f(x, x') = 1$ for all x . Two experiments π and π' are equivalent, denoted by $\pi \sim \pi'$, if $\pi \trianglelefteq \pi'$ and $\pi' \trianglelefteq \pi$.*

Note that this definition provides a different interpretation of $\Phi(\pi)$ as the set of garbles π' of π such that $|X'| \leq |A|$. For each function f satisfying the conditions of the definition, I will denote by $f \circ \pi$ the corresponding garble of π .

It is useful to prove a few basic results about garbles. The first of these results shows that, if one can observe two experiments $f_1 \circ \pi$ with realization space X_1 , and $f_2 \circ \pi$ with realization space X_2 , which are both garbles of π , then the experiment $f_1 f_2 \circ \pi$, with realization space $X_1 \times X_2$, is also a garble of π . This easily extends to a finite number of garbles.

Lemma 1. *Let $f_1 \circ \pi, \dots, f_K \circ \pi$ be garbles of π . Then*

$$f_1 \cdots f_K \circ \pi \trianglelefteq \pi.$$

Proof. Let $g(x, x_1, \dots, x_K) = f_1(x, x_1) \cdots f_K(x, x_K)$. Then

$$\sum_{x_1, \dots, x_K} g(x, x_1, \dots, x_K) = \sum_{x_1} f_1(x, x_1) \cdots \sum_{x_K} f_K(x, x_K) = 1.$$

so that $f_1 \cdots f_K \circ \pi$ is indeed a garble of π . □

The second result shows that if one is allowed to combine experiments from the set of binary garbles of π , i.e. garbles with support of size 2, then one can reconstitute all the information in π . The idea is simple: for each possible realization x of π , define the binary garble of π that returns 1 if x is realized and 0 otherwise; then combining all these garbles gives exactly the same information as π . Without loss of generality, I can fix the set $B = \{0, 1\}$, and denote the set of binary garbles of π by

$$\Gamma_b(\pi) = \left\{ \pi'(x', \omega) : \exists f : X \times B \rightarrow \mathbb{R}^+, \pi'(x', \omega) = \sum_x f(x, x')\pi(x, \omega) \text{ and } \sum_{x' \in B} f(x, x') = 1 \right\}.$$

Lemma 2 (Reconstitution from Binary Garbles). *Consider an experiment π with support X . Then there exists $|X|$ binary garbles $f_1 \circ \pi, \dots, f_{|X|} \circ \pi \in \Gamma_b(\pi)$ such that*

$$f_1 \cdots f_{|X|} \circ \pi \sim \pi.$$

Proof. Let $x_1, \dots, x_{|X|}$ be the elements of X . Then let $f_k(x, 1) = \mathbb{1}_{x=x_k}$ and $f_k(x, 0) = 1 - f_k(x, 1)$. The $|X|$ functions thus defined satisfy the conditions of [Definition 2](#), so they generate $|X|$ binary garbles $f_1 \circ \pi, \dots, f_{|X|} \circ \pi$. It is easy to see that $(f_k \circ \pi)(1, \omega) = \pi(x_k, \omega)$, therefore observing the combined outcomes of all the experiments $f_1 \circ \pi, \dots, f_{|X|} \circ \pi$ intuitively allows to reconstitute the π . To show this formally consider the experiment $f_1 \cdots f_{|X|} \circ \pi$. Its realization space is $\{0, 1\}^{|X|}$, but in fact the only vectors that occur with positive probability are the vectors with 0 on every dimension except one. Let e^k be the vector with a 1 on the k -th dimension and zeros elsewhere. Then for every $k = 1, \dots, |X|$

$$(f_1 \cdots f_{|X|} \circ \pi)(e^k, \omega) = \pi(x_k, \omega),$$

which proves that $f_1 \cdots f_{|X|} \circ \pi \sim \pi$. In fact, $f_1 \cdots f_{|X|} \circ \pi$ is exactly π , up to a recoding of the set X . □

3 Blackwell's Theorem

Blackwell's theorem says that the usefulness order and the garbling order are the same. I decompose the proof in two steps. First, I show by classical arguments that an experiment is more informative than another if and only if it generates a larger policy space in the set containment order. Second, I show that an experiment generates a larger policy space than another one if and only if the latter is a garbling of the former. The latter part is the novel one and it relies on the binary decomposition result. The idea for the difficult implication is to show that, if π leads to a larger policy space than π' , then the binary reconstitution of π' , which is informationally equivalent to π' , is a garble of π .

Theorem 1 (Blackwell). $\pi \succeq \pi' \Leftrightarrow \Phi(\pi) \supseteq \Phi(\pi') \Leftrightarrow \pi \supseteq \pi'$.

Proof. I write one lemma for each step.

Lemma 3. $\pi \succeq \pi' \Leftrightarrow \Phi(\pi) \supseteq \Phi(\pi')$

Proof. \Leftarrow is due to the fact that maximizing a function over a larger set always yields a higher value. \Rightarrow is due to a separation theorem. Indeed, suppose that there exists a policy $\varphi(a, \omega)$ in $\Phi(\pi') \setminus \Phi(\pi)$. Then because $\Phi(\pi)$ is a closed convex set, the hyperplane separation theorem implies the existence of a vector $u \in U$ such that $\sum_{a, \omega} u(a, \omega) \varphi(a, \omega) > V(\pi, u)$. \square

Lemma 4. $\Phi(\pi) \supseteq \Phi(\pi') \Leftrightarrow \pi \supseteq \pi'$

Proof. \Leftarrow is the more natural sense. Suppose that π' is a garble of π , and let $f(\cdot)$ be the associated garbling function. Let $\varphi \in \Phi(\pi')$ be the policy generated by the associated strategy $\sigma \in \Sigma(\pi')$. Consider the strategy

$$\hat{\sigma}(a|x) \equiv \sum_{x'} f(x, x') \sigma(a|x').$$

It is an element of $\Sigma(\pi)$ since

$$\sum_a \hat{\sigma}(a|x) = \sum_{x'} f(x, x') \sum_a \sigma(a|x') = 1.$$

And I can write

$$\begin{aligned} \varphi(a, \omega) &= \sum_{x'} \sigma(a|x') \pi'(x', \omega) \\ &= \sum_{x'} \sigma(a|x') \sum_x f(x, x') \pi(x, \omega) \\ &= \sum_x \sum_{x'} f(x, x') \sigma(a|x') \pi(x, \omega) \\ &= \sum_x \hat{\sigma}(a|x) \pi(x, \omega), \end{aligned}$$

which shows that $\varphi \in \Phi(\pi)$.

For \Rightarrow , suppose $\Phi(\pi') \subseteq \Phi(\pi)$. Then, since $|A| \geq 2$, I have $\Gamma_b(\pi') \subseteq \Phi(\pi') \subseteq \Phi(\pi)$. Then, by Lemma 2, I can pick $|X'|$ binary garbles $f_1 \circ \pi', \dots, f_{|X'|} \circ \pi'$ in $\Phi(\pi')$ that reconstitute π' , so that $f_1 \cdots f_{|X'|} \circ \pi' \sim \pi'$.

Since $f_k \circ \pi' \in \Phi(\pi)$, $f_k \circ \pi'$ is a garble of π , so it is possible to find a function $g_k : X \times B \rightarrow [0, 1]$ such that $g_k(x, 0) + g_k(x, 1) = 1$, and $f_k \circ \pi' = g_k \circ \pi$.

Consider the function $g : X \times B \rightarrow \mathbb{R}^+$ defined by $g(x, b) = \sum_k g_k(x, b)$. I can write

$$\begin{aligned}
\boxed{\sum_x g(x, 1)\pi(x, \omega)} &= \sum_x \sum_k g_k(x, 1)\pi(x, \omega) = \sum_k \sum_x g_k(x, 1)\pi(x, \omega) \\
&= \sum_k (\pi \circ g_k)(1, \omega) = \sum_k (\pi' \circ f_k)(1, \omega) \\
&= \sum_k \pi'(x_k, \omega) = p(\omega) \\
&= \boxed{\sum_x \pi(x, \omega)}
\end{aligned}$$

This can be seen as a system of $|\Omega|$ equations in $|X|$ unknowns, the $(g(x, 1))_{x \in X}$. It has at least one solution which is $g(x, 1) = 1$, for all x . If $|X| \leq |\Omega|$, this is the unique solution, and therefore the functions $g_k(\cdot)$ must be such that $g(x, 1) = 1$, for all x .

Suppose instead that $|X| > |\Omega|$. Then I show that the $g_k(\cdot)$ functions can be chosen so that $g(x, 1) = 1$ for all x . To see this note first that because $g_k(x, 1) + g_k(x, 0) = 1$, the problem of finding the $g_k(\cdot)$ functions can be reduced to solving the system of $|X'| \times |\Omega|$ equations in $|X'| \times |X|$ unknowns given by $\sum_x \in X g_k(x, 1)\pi(x, \omega) = \pi'(x, \omega)$, for each $k = 1, \dots, |X'|$, and each $\omega \in \Omega$. Because $|X| > |\Omega|$ the system has multiple solutions. Adding the $|X|$ equations $g(x, 1) = \sum_k g_k(x, 1) = 1$ to the system leaves the number of equation below the number of unknowns, so we can indeed choose the $g_k(\cdot)$ functions so that $g(x, 1) = 1$, for all x .

Knowing this, I show that $f_1 \cdots f_{|X'|} \circ \pi'$ is a garble of π as follows. For every $e \in \{0, 1\}^{|X'|}$,

I have

$$\begin{aligned}
(\pi' \circ f_1 \cdots f_{|X'|}) (e, \omega) &= \sum_k \mathbb{1}_{e=e^k} (\pi' \circ f_k) (1, \omega) \\
&= \sum_k \mathbb{1}_{e=e^k} (\pi \circ g_k) (1, \omega) \\
&= \sum_k \mathbb{1}_{e=e^k} \sum_x g_k(x, 1) \pi(x, \omega) \\
&= \sum_x \underbrace{\sum_k \mathbb{1}_{e=e^k} g_k(x, 1)}_{\equiv h(x,e)} \pi(x, \omega)
\end{aligned}$$

Hence, to prove that $f_1 \cdots f_{|X'|} \circ \pi'$ is a garble of π , I just need to show that $\sum_e h(x, e) = 1$. To see this note that $h(x, e) = 0$ if e is not one of the e^k vectors, and $h(x, e^k) = g_k(x, 1)$. Therefore,

$$\sum_e h(x, e) = \sum_k g_k(x, 1) = g(x, 1) = 1.$$

□

□

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