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# A PROOF OF BLACKWELL'S THEOREM 

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# A Proof of Blackwell's Theorem* 

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#### Abstract

This note gives a new proof of Blackwell's celebrated result. The result is a bit stronger than the classical version since the action set and the prior are fixed, and only the utility of the decision maker varies. I show directly that a decision maker has access to a larger set of joint distributions over actions and states of the world if and only if her information improves in the garbling order.


## 1 Introduction

This note provides a proof of Blackwell's theorem (Blackwell, 1951, 1953). If a decision maker is identified with a prior on the states of the world, an action set, and a utility function over actions and states of the world, Blackwell's theorem says that an experiment $\pi$, that provides information about the state of the world, is preferred by every decision maker to an experiment $\pi^{\prime}$ if and only if $\pi^{\prime}$ is a garble of $\pi$. The proof I provide is relatively simple, and has the merit of making the intuitive point that the choice set of the decision maker is enlarged by moving from $\pi^{\prime}$ to $\pi$ if and only if $\pi^{\prime}$ is a garble of $\pi$, which is absent from other proofs (Blackwell, 1951, 1953; Ponssard, 1975; Cremer, 1982; Leshno and Spector, 1992). Another advantage of this proof is that it varies only the utility of the decision maker, and not the prior or the action

[^0]set ${ }^{1}$, so the difficult direction of the result ( $\pi$ more useful than $\pi^{\prime}$ implies that $\pi^{\prime}$ is a garble of $\pi)$ is slightly stronger here than in Blackwell's original formulation.

## 2 Setup and Preliminary Results

There is a finite action set $A$, with $|A| \geq 2$, and a finite set of states of the world $\Omega$. The prior is a probability distribution $p(\omega)$ in $\Delta(\Omega)$. The payoff of the decision maker is given by a real valued payoff function $u(a, \omega)$. Let $U$ be the set of such payoff functions. An experiment is given by a random variable $x$, with finite support $X$ and a joint distribution function $\pi$ on $X \times \Omega$ with marginal $p(\cdot)$ on $\Omega$. When the decision maker can observe the realization of $x$, but not that of $\omega$, she has access to mixed strategies $\sigma(a \mid x)$, with $\sum_{a} \sigma(a \mid x)=1$ for all $x$. Let $\Sigma(\pi)$ be the set of strategies accessible to a decision maker endowed with experiment $\pi$. Ultimately, the decision maker only cares about the joint distributions of actions and states of the world, $\varphi(a, \omega)$. Let $\Phi(\pi)$ be the set of joint distributions she can generate when endowed with $\pi$, or policy space. It is restricted by her lack of knowledge in the following way:

$$
\Phi(\pi)=\left\{\varphi(a, \omega): \exists \sigma \in \Sigma(\pi), \varphi(a, \omega)=\sum_{x} \sigma(a \mid x) \pi(x, \omega)\right\} .
$$

It is easy to show that this set is a compact, and convex subset of $[0,1]^{|A| \times|\Omega|}$. Then the problem of decision maker $u$ endowed with experiment $\pi$ is given by the following linear program

$$
V(\pi, u)=\max _{\varphi \in \Phi(\pi)} \sum_{a, \omega} \varphi(a, \omega) u(a, \omega) .
$$

Definition 1 (Usefulness Order). I say that an experiment $\pi$ is more useful than another experiment $\pi^{\prime}$, and write $\pi \succeq \pi^{\prime}$, if all decision makers get a higher value when endowed with $\pi$ than when they are endowed with $\pi^{\prime}$, that is,

$$
\pi \succeq \pi^{\prime} \Leftrightarrow V(\pi, u) \geq V\left(\pi^{\prime}, u\right), \forall u \in U
$$

[^1]Definition 2 (Garbling Order). I say that $\pi^{\prime}$ is a garble of $\pi$, and write $\pi^{\prime} \unlhd \pi$ if there exists a function $f: X \times X^{\prime} \rightarrow[0,1]$ such that $\pi^{\prime}\left(x^{\prime}, \omega\right)=\sum_{x} f\left(x, x^{\prime}\right) \pi(x, \omega)$ and $\sum_{x^{\prime}} f\left(x, x^{\prime}\right)=1$ for all $x^{\prime}$. Two experiments $\pi$ and $\pi^{\prime}$ are equivalent, denoted by $\pi \sim \pi^{\prime}$, if $\pi \unlhd \pi^{\prime}$ and $\pi^{\prime} \unlhd \pi$.

Note that this definition provides a different interpretation of $\Phi(\pi)$ as the set of garbles $\pi^{\prime}$ of $\pi$ such that $\left|X^{\prime}\right| \leq|A|$. For each function $f$ satisfying the conditions of the definition, I will denote by $f \circ \pi$ the corresponding garble of $\pi$.

It is useful to prove a few basic results about garbles. The first of these results shows that, if one can observe two experiments $f_{1} \circ \pi$ with realization space $X_{1}$, and $f_{2} \circ \pi$ with realization space $X_{2}$, which are both garbles of $\pi$, then the experiment $f_{1} f_{2} \circ \pi$, with realization space $X_{1} \times X_{2}$, is also a garble of $\pi$. This easily extends to a finite number of garbles.

Lemma 1. Let $f_{1} \circ \pi, \cdots, f_{K} \circ \pi$ be garbles of $\pi$. Then

$$
f_{1} \cdots f_{k} \circ \pi \unlhd \pi
$$

Proof. Let $g\left(x, x_{1}, \cdots, x_{K}\right)=f_{1}\left(x, x_{1}\right) \cdots f_{2}\left(x, x_{K}\right)$. Then

$$
\sum_{x_{1}, \cdots, x_{K}} g\left(x, x_{1}, \cdots, x_{K}\right)=\sum_{x_{1}} f_{1}\left(x, x_{1}\right) \cdots \sum_{x_{K}} f_{K}\left(x, x_{K}\right)=1 .
$$

so that $f_{1} \cdots f_{K} \circ \pi$ is indeed a garble of $\pi$.
The second result shows that if one is allowed to combine experiments from the set of binary garbles of $\pi$, i.e. garbles with support of size 2, then one can reconstitute all the information in $\pi$. The idea is simple: for each possible realization $x$ of $\pi$, define the binary garble of $\pi$ that returns 1 if $x$ is realized and 0 otherwise; then combining all these garbles gives exactly the same information as $\pi$. Without loss of generality, I can fix the set $B=\{0,1\}$, and denote the set of binary garbles of $\pi$ by
$\Gamma_{b}(\pi)=\left\{\pi^{\prime}\left(x^{\prime}, \omega\right): \exists f: X \times B \rightarrow \mathbb{R}^{+}, \pi^{\prime}\left(x^{\prime}, \omega\right)=\sum_{x} f\left(x, x^{\prime}\right) \pi(x, \omega)\right.$ and $\left.\sum_{x^{\prime} \in B} f\left(x, x^{\prime}\right)=1\right\}$.

Lemma 2 (Reconstitution from Binary Garbles). Consider an experiment $\pi$ with support $X$. Then there exists $|X|$ binary garbles $f_{1} \circ \pi, \cdots, f_{|X|} \circ \pi \in \Gamma_{b}(\pi)$ such that

$$
f_{1} \cdots f_{|X|} \circ \pi \sim \pi
$$

Proof. Let $x_{1}, \cdots, x_{|X|}$ be the elements of $X$. Then let $f_{k}(x, 1)=\mathbb{1}_{x=x_{k}}$ and $f_{k}(x, 0)=1-$ $f_{k}(x, 1)$. The $|X|$ functions thus defined satisfy the conditions of Definition 2, so they generate $|X|$ binary garbles $f_{1} \circ \pi, \cdots, f_{|X|} \circ \pi$. It is easy to see that $\left(f_{k} \circ \pi\right)(1, \omega)=\pi\left(x_{k}, \omega\right)$, therefore observing the combined outcomes of all the experiments $f_{1} \circ \pi, \cdots, f_{|X|} \circ \pi$ intuitively allows to reconstitute the $\pi$. To show this formally consider the experiment $f_{1} \cdots f_{|X|} \circ \pi$. Its realization space is $\{0,1\}^{|X|}$, but in fact the only vectors that occur with positive probability are the vectors with 0 on every dimension except one. Let $e^{k}$ be the vector with a 1 on the $k$-th dimension and zeros elsewhere. Then for every $k=1, \cdots,|X|$

$$
\left(f_{1} \cdots f_{|X|} \circ \pi\right)\left(e^{k}, \omega\right)=\pi\left(x_{k}, \omega\right)
$$

which proves that $f_{1} \cdots f_{|X|} \circ \pi \sim \pi$. In fact, $f_{1} \cdots f_{|X|} \circ \pi$ is exactly $\pi$, up to a recoding of the set $X$.

## 3 Blackwell's Theorem

Blackwell's theorem says that the usefulness order and the garbling order are the same. I decompose the proof in two steps. First, I show by classical arguments that an experiment is more informative than another if and only if it generates a larger policy space in the set containment order. Second, I show that an experiment generates a larger policy space than another one if and only if the latter is a garbling of the former. The latter part is the novel one and it relies on the binary decomposition result. The idea for the difficult implication is to show that, if $\pi$ leads to a larger policy space than $\pi^{\prime}$, then the binary reconstitution of $\pi^{\prime}$, which is informationally equivalent to $\pi^{\prime}$, is a garble of $\pi$.

Theorem 1 (Blackwell). $\pi \succeq \pi^{\prime} \Leftrightarrow \Phi(\pi) \supseteq \Phi\left(\pi^{\prime}\right) \Leftrightarrow \pi \unrhd \pi^{\prime}$.

Proof. I write one lemma for each step.
Lemma 3. $\pi \succeq \pi^{\prime} \Leftrightarrow \Phi(\pi) \supseteq \Phi\left(\pi^{\prime}\right)$

Proof. $\Leftarrow$ is due to the fact that maximizing a function over a larger set always yields a higher value. $\Rightarrow$ is due to a separation theorem. Indeed, suppose that there exists a policy $\varphi(a, \omega)$ in $\Phi\left(\pi^{\prime}\right) \backslash \Phi(\pi)$. Then because $\Phi(\pi)$ is a closed convex set, the hyperplane separation theorem implies the existence of a vector $u \in U$ such that $\sum_{a, \omega} u(a, \omega) \varphi(a, \omega)>V(\pi, u)$.

Lemma 4. $\Phi(\pi) \supseteq \Phi\left(\pi^{\prime}\right) \Leftrightarrow \pi \unrhd \pi^{\prime}$

Proof. $\Leftarrow$ is the more natural sense. Suppose that $\pi^{\prime}$ is a garble of $\pi$, and let $f(\cdot)$ be the associated garbling function. Let $\varphi \in \Phi\left(\pi^{\prime}\right)$ be the policy generated by the associated strategy $\sigma \in \Sigma\left(\pi^{\prime}\right)$. Consider the strategy

$$
\hat{\sigma}(a \mid x) \equiv \sum_{x^{\prime}} f\left(x, x^{\prime}\right) \sigma\left(a \mid x^{\prime}\right) .
$$

It is an element of $\Sigma(\pi)$ since

$$
\sum_{a} \hat{\sigma}(a \mid x)=\sum_{x^{\prime}} f\left(x, x^{\prime}\right) \sum_{a} \sigma\left(a \mid x^{\prime}\right)=1
$$

And I can write

$$
\begin{aligned}
\varphi(a, \omega) & =\sum_{x^{\prime}} \sigma\left(a \mid x^{\prime}\right) \pi^{\prime}\left(x^{\prime}, \omega\right) \\
& =\sum_{x^{\prime}} \sigma\left(a \mid x^{\prime}\right) \sum_{x} f\left(x, x^{\prime}\right) \pi(x, \omega) \\
& =\sum_{x} \sum_{x^{\prime}} f\left(x, x^{\prime}\right) \sigma\left(a \mid x^{\prime}\right) \pi(x, \omega) \\
& =\sum_{x} \hat{\sigma}(a \mid x) \pi(x, \omega),
\end{aligned}
$$

which shows that $\varphi \in \Phi(\pi)$.

For $\Rightarrow$, suppose $\Phi\left(\pi^{\prime}\right) \subseteq \Phi(\pi)$. Then, since $|A| \geq 2$, I have $\Gamma_{b}\left(\pi^{\prime}\right) \subseteq \Phi\left(\pi^{\prime}\right) \subseteq \Phi(\pi)$. Then, by Lemma 2, I can pick $\left|X^{\prime}\right|$ binary garbles $f_{1} \circ \pi^{\prime}, \cdots f_{\left|X^{\prime}\right|} \circ \pi^{\prime}$ in $\Phi\left(\pi^{\prime}\right)$ that reconstitute $\pi^{\prime}$, so that $f_{1} \cdots f_{\left|X^{\prime}\right|} \circ \pi^{\prime} \sim \pi^{\prime}$.

Since $f_{k} \circ \pi^{\prime} \in \Phi(\pi), f_{k} \circ \pi^{\prime}$ is a garble of $\pi$, so it is possible to find a function $g_{k}: X \times B \rightarrow$ $[0,1]$ such that $g_{k}(x, 0)+g_{k}(x, 1)=1$, and $f_{k} \circ \pi^{\prime}=g_{k} \circ \pi$.

Consider the function $g: X \times B \rightarrow \mathbb{R}^{+}$defined by $g(x, b)=\sum_{k} g_{k}(x, b)$. I can write

$$
\begin{aligned}
\sum_{x} g(x, 1) \pi(x, \omega) & =\sum_{x} \sum_{k} g_{k}(x, 1) \pi(x, \omega)=\sum_{k} \sum_{x} g_{k}(x, 1) \pi(x, \omega) \\
& =\sum_{k}\left(\pi \circ g_{k}\right)(1, \omega)=\sum_{k}\left(\pi^{\prime} \circ f_{k}\right)(1, \omega) \\
& =\sum_{k} \pi^{\prime}\left(x_{k}, \omega\right)=p(\omega) \\
& =\sum_{x} \pi(x, \omega)
\end{aligned}
$$

This can be seen as a system of $|\Omega|$ equations in $|X|$ unknowns, the $(g(x, 1))_{x \in X}$. It has at least one solution which is $g(x, 1)=1$, for all $x$. If $|X| \leq|\Omega|$, this is the unique solution, and therefore the functions $g_{k}(\cdot)$ must be such that $g(x, 1)=1$, for all $x$.

Suppose instead that $|X|>|\Omega|$. Then I show that the $g_{k}(\cdot)$ functions can be chosen so that $g(x, 1)=1$ for all $x$. To see this note first that because $g_{k}(x, 1)+g_{k}(x, 0)=1$, the problem of finding the $g_{k}(\cdot)$ functions can be reduced to solving the system of $\left|X^{\prime}\right| \times|\Omega|$ equations in $\left|X^{\prime}\right| \times|X|$ unknowns given by $\sum_{x} \in X g_{k}(x, 1) \pi(x, \omega)=\pi^{\prime}(x, \omega)$, for each $k=1, \ldots,\left|X^{\prime}\right|$, and each $\omega \in \Omega$. Because $|X|>|\Omega|$ the system has multiple solutions. Adding the $|X|$ equations $g(x, 1)=\sum_{k} g_{k}(x, 1)=1$ to the system leaves the number of equation below the number of unknowns, so we can indeed choose the $g_{k}(\cdots)$ functions so that $g(x, 1)=1$, for all $x$.

Knowing this, I show that $f_{1} \cdots f_{\left|X^{\prime}\right|} \circ \pi^{\prime}$ is a garble of $\pi$ as follows. For every $e \in\{0,1\}^{\left|X^{\prime}\right|}$,

I have

$$
\begin{aligned}
\left(\pi^{\prime} \circ f_{1} \cdots f_{\left|X^{\prime}\right|}\right)(e, \omega) & =\sum_{k} \mathbb{1}_{e=e^{k}}\left(\pi^{\prime} \circ f_{k}\right)(1, \omega) \\
& =\sum_{k} \mathbb{1}_{e=e^{k}}\left(\pi \circ g_{k}\right)(1, \omega) \\
& =\sum_{k} \mathbb{1}_{e=e^{k}} \sum_{x} g_{k}(x, 1) \pi(x, \omega) \\
& =\sum_{x} \underbrace{\sum_{k} \mathbb{1}_{e=e^{k}} g_{k}(x, 1) \pi(x, \omega)}_{\equiv h(x, e)}
\end{aligned}
$$

Hence, to prove that $f_{1} \cdots f_{\left|X^{\prime}\right|} \circ \pi^{\prime}$ is a garble of $\pi$, I just need to show that $\sum_{e} h(x, e)=1$. To see this note that $h(x, e)=0$ if $e$ is not one of the $e^{k}$ vectors, and $h\left(x, e^{k}\right)=g_{k}(x, 1)$. Therefore,

$$
\sum_{e} h(x, e)=\sum_{k} g_{k}(x, 1)=g(x, 1)=1 .
$$

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[^1]:    ${ }^{1}$ Leshno and Spector (1992) also fix the action set, but use a different proof technique based on matrices.

