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# AN EMPIRICAL FRAMEWORK FOR MATCHING WITH IMPERFECTLY TRANSFERABLE UTILITY

ALFRED GALICHON<sup>†</sup>, SCOTT DUKE KOMINERS<sup>‡</sup>, AND SIMON WEBER<sup>§</sup>

**ABSTRACT.** We introduce an empirical framework for models of matching with imperfectly transferable utility and unobserved heterogeneity in tastes. Our framework includes as special cases the classic fully- and non-transferable utility models, collective models, and settings with taxes on transfers, deadweight losses, and risk aversion. We characterize equilibrium and conditions for identification, and derive comparative statics.

**Keywords:** Sorting, Matching, Marriage Market, Becker–Coase Theorem, Intrahousehold Allocation, Imperfectly Transferable Utility.

**JEL Classification:** C78, D3, J21, J23, J31, J41.

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## 1. INTRODUCTION

In the classic model of matching with Nontransferable Utility (Gale and Shapley (1962)), there is no possibility of compensating transfer between partners: if a man of type  $x$  marries a woman of type  $y$ , the man receives utility  $\alpha_{xy}$  and the woman receives utility  $\gamma_{xy}$ , without the possibility of transferring any utility across the pair. With Transferable Utility (Becker (1973), Shapley and Shubik (1972)), by contrast, there is a numeraire good which is freely transferable across partners. In this case, a man and woman who match may decide on a transfer  $t_{x\leftarrow y}$  (possibly negative) from the woman to the man, so that the man's equilibrium utility is  $\alpha_{xy} + t_{x\leftarrow y}$  and the woman's equilibrium utility is  $\gamma_{xy} - t_{x\leftarrow y}$ . Here, the transfer technology is assumed frictionless, so that so that the couple's sum of the utilities is  $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$ , independent of the transfer.

Sometimes, the transfer technology (or lack thereof) is clear from the market context: In school choice and organ exchanges, for example, transfers are often explicitly forbidden; hence, NTU matching seems a natural model in those settings. TU matching, meanwhile, seems to be a good first approximation for describing labor markets, trading networks, and demand for quality, as in all these settings there is a market-clearing price.

In many markets, however, there can be frictions that partially impede the transfer of utility between matched partners. This possibility seems natural in marriage markets, where the transfers between partners might take the form of favor exchange (rather than cash), and the cost of a favor to one partner may not exactly equal the benefit to the other. Frictions are also present in nearly every labor market—because of taxation, an employer must pay more in wages than its employees actually receive (Jaffe and Kominers (2014)). A market with transfer frictions has *Imperfectly Transferable Utility* (ITU): although some utility can be transferred among partners, the transfer technology is imperfect.

In this paper, we develop an empirical framework for matching models with Imperfectly Transferable Utility (ITU), using a structure of heterogeneities in preferences à-la Choo and Siow (2006) and Galichon and Salanié (2014). Our setting encompasses (as limits) both

the classic Transferable Utility (TU) and Nontransferable Utility (NTU) models,<sup>1</sup> as well as collective models and settings with taxation of transfers. In contrast with the existing literature on ITU matching models, our setting allows for a very simple characterization of equilibrium, as well as its computation and analysis. We prove existence and uniqueness of the equilibrium solution in a general ITU model with logit heterogeneities. Then, we find conditions under which the affinity parameters and the size of transfer frictions are simultaneously identified. Finally, we present comparative statics results and illustrate how the presence of transfer frictions can reverse classical policy intuitions from the TU model.

Our approach allows us to tackle two challenges, which have been difficult to address using previous models: First, we can make realistic predictions of the welfare consequences of intra-household pre-equilibrium transfers, moving beyond the Becker–Coase Theorem. Second, we can integrate the literature on collective models into matching, providing a model of endogenous determination of the collective sharing rule using the matching-theoretic *stability* solution concept.

**Beyond the Becker–Coase Theorem.** In the family economics literature, the Coase (1960) Theorem takes the following form due to Becker (1991):

“although divorce might seem more difficult when mutual consent is required than when either alone can divorce at will, the frequency and incidence of divorce should be similar with these or other rules if couples contemplating divorce can easily bargain with each other [that is, if utility is transferable].”

This stark result is often called the Becker–Coase Theorem (see Becker et al. (1977)). In essence, the Becker–Coase Theorem states that if some policy changes men’s and women’s match affinities  $\alpha_{xy}$  and  $\gamma_{xy}$  to  $\alpha_{xy} - s$  and  $\gamma_{xy} + s$  (for instance, by affecting the legal framework of divorce), in such a way that the joint affinity remains equal to  $\alpha_{xy} + \gamma_{xy}$ , then the number of matches  $\mu_{xy}$  between men of type  $x$  and women of type  $y$  will be unaffected. The Becker–Coase Theorem implies that equilibrium transfers will adjust so that the matching patterns and the welfare remain the same before and after the change

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<sup>1</sup>The NTU matching arises as the limit as transfer frictions grow arbitrarily large; TU matching arises as the frictions vanish.

in legislation.<sup>2</sup> More generally, it is a basic implication of TU models that equilibrium quantities (number of matches and equilibrium payoffs) only depend on the joint surplus  $\alpha_{xy} + \gamma_{xy}$  generated within each pair. The Becker–Coase Theorem’s conclusion, however, depends crucially on the TU modeling assumption: when this assumption is not made, that is, when transfers are either imperfect or impossible, even “zero-sum” policies that just shift match affinity from one side of the market to the other can have adverse welfare consequences, making agents on *both* sides of the market worse off in equilibrium. The present paper, with the comparative static results it offers, is a far-reaching extension of this type of question. Namely, we will investigate the effect of a change in the structural parameters (number of individuals of each type, affinity parameters, or transfer friction parameter) on the equilibrium outcome (matching patterns and post-transfer individual surpluses).

**Endogenous Sharing Rules.** The field of family economics is mostly split between the matching approach of Becker (1973), which focuses on matching patterns and the sharing of the surplus in a TU model, and the collective approach of Chiappori (1991), which focuses on intra-household bargaining over a potentially complex feasible utility set, generally falling under the ITU framework, and only seldom amenable to a TU model. Due to this discrepancy in the underlying settings, the two approaches have not yet been embedded in a single empirical framework. This is illustrated by Choo and Siow’s (2006) contention, stated in their conclusion, that “[their] model of marriage matching should be integrated with models of intrahousehold allocations such as those of Lundberg and Pollak (1993) and Chiappori et al. (2002).” Merging marriage matching models with intra-household bargaining models has been difficult in the past because the models (like that of Choo and Siow (2006)) based on the Becker (1973) framework assume transferable utility, whereas most models of intra-household allocation cannot be expressed as TU models because of

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<sup>2</sup>These strong implications are often not verified by the data. For example, Lundberg et al. (1997), Friedberg (1998), Wolfers (2006), Halla (2013), and Fernández and Wong (2014) exploit changes in divorce legislation (unilateral divorce, joint custody, etc), and find significant effects on divorce rates.

inefficiencies in the bargaining process.<sup>3</sup> Our ITU matching framework allows us to model marriage matching and intra-household bargaining simultaneously: under ITU matching, while matched partners can bargain with each other, intra-household bargaining involves costs or frictions. As a result, we are able to endogenize the “sharing rule” used in the collective approach, which is the set of weights on agents’ utilities that determine the selection of an outcome on the Pareto frontier.

**Relation to the Literature.** The theory of ITU matching has been studied by a number of authors: Crawford and Knoer (1981), Kelso and Crawford (1982), and Hatfield and Milgrom (2005) found conditions and algorithms for finding competitive equilibrium/stable outcomes in ITU matching markets, and analyzed the structure of the set of equilibria. Kaneko (1982), Quinzii (1984), Alkan (1988) Alkan and Gale (1988), Gale (1984), Demange and Gale (1985) provided results on the existence of equilibrium and study properties of the core. Jaffe and Kominers (2014) studied the problem of matching with linear and lump-sum taxes and provided comparative statics results. Legros and Newman (2007) found conditions under which positive assortativeness arises in ITU models; they applied these findings to problems of matching under uncertainty with risk aversion. Chiappori (2012) provided an illustrative example. Chade and Eeckhout (2014) extended the Legros and Newman (2007) analysis to the case where agents have different risky endowments. Samuelson and Nöldeke (2014) used an ITU framework in their analysis of pre-matching investment. Azevedo and Leshno (2014) provide existence and comparative statics in an NTU model with a continuum of types.

However, the literature on the structural estimation of matching models has so far, to the best of our knowledge, been restricted to the TU case and NTU cases. In the wake of the seminal work by Choo and Siow (2006), many papers have exploited heterogeneity in preferences for identification. In the TU case, see Fox (2010), Chiappori, Oreffice and Quintana-Domeque (2012), Galichon and Salanié (2014), Chiappori, Salanié, Weiss (2014), Dupuy and Galichon (2014), and Jacquemet and Robin (2014). In the NTU case, we refer

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<sup>3</sup>There are exceptions—see, e.g., the work of Bowning et al. (2014, pp. 83 and 118), in which one private good is assumed to provide the same marginal utility to both members of the household, and thus can be used to transfer utility without friction.

to Dagsvik (2000), Menzel (2014), Hitsch, Hortacısu, and Ariely (2010), and Agarwal (2014). To the best of our knowledge, our work is the first to provide an empirical framework for general ITU models.

**Organization of the Paper.** The remainder of the paper is organized as follows. Section 2 lays out our empirical framework of the paper, and determines the equations characterizing equilibrium. Section 4 deals with the empirically important case where the heterogeneity in taste has a logit structure; in the logit case, we prove existence and uniqueness of the equilibrium, provide a computationally efficient algorithm for finding equilibria, and discuss identification and estimation. In Section 5, we derive comparative statics and conduct welfare analysis in the context of the Becker–Coase Theorem. Section 6 concludes. All proofs are presented in the Appendix.

## 2. FRAMEWORK

**2.1. Basic model.** We begin by modeling a small-population, two-sided matching framework with imperfect transfers (and without heterogeneity): There is a set  $I$  of *men* and a set  $J$  of *women*; man  $i \in I$  receives match utility (which we call *affinity*)  $\alpha_{ij}$  for matching with woman  $j \in J$ , while woman  $j$  receives match utility (affinity)  $\gamma_{ij}$  for matching with man  $i$ . If man  $i$  and woman  $j$  match, they split the joint affinity generated by the match,  $\alpha_{ij} + \gamma_{ij}$ , under the constraint that the post-bargaining utility of the man, denoted  $u_i$ , and the post-bargaining utility of the woman, denoted  $v_j$ , must satisfy the feasibility constraint

$$\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) \leq 0, \quad (2.1)$$

where the transfer function  $\Psi$  is assumed continuous and isotone with respect to its arguments. We may think of

$$t_{i \leftarrow j} := u_i - \alpha_{ij} \text{ and } t_{j \leftarrow i} = v_j - \gamma_{ij} \quad (2.2)$$

as representing the *transfers* received by  $i$  and  $j$ , respectively, to bring their individual surpluses up to  $u_i$  and  $v_j$ . Thus, the transfer function  $\Psi$  determines the (possibly pair-specific) frictions in within-pair transfers. As we detail in the next section, our setting

embeds both the standard NTU matching model (Gale and Shapley, 1962), and the standard TU model (Koopmans and Beckman, 1957; Shapley and Shubik, 1971; Becker, 1973).

We denote by  $u_i$  (resp.  $v_j$ ) the (equilibrium) outcome surplus of man  $i$  (resp. woman  $j$ ). Now, if  $\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) < 0$ , then there exist  $u'_i$  and  $v'_j$  so that  $u_i < u'_i$ ,  $v_j < v'_j$ , and  $\Psi_{ij}(u'_i - \alpha_{ij}, v'_j - \gamma_{ij}) \leq 0$ , so that  $i$  and  $j$  can improve their private outcomes by matching together.

Outcome *stability* implies that  $\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) \geq 0$ , with equality if  $i$  and  $j$  are matched. Formally, a (stable) *equilibrium* is a triple  $(\mu, u, v)$  such that:

- (1) for all  $i$  and  $j$ ,  $\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) \geq 0$ ;
- (2) if  $\mu_{ij} = 1$ , then  $\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) = 0$ ;
- (3)  $\sum_j \mu_{ij} \leq 1$  and  $\sum_i \mu_{ij} \leq 1$ ; and
- (4)  $\mu_{ij} \in \{0, 1\}$ .

In the sequel we shall refer to  $\alpha_{xy}$  and  $\gamma_{xy}$  as *affinity terms*, and to  $\Psi_{xy}$  as the *transfer function*.

**2.2. Example Specifications.** Now, we provide examples of specifications of the transfer function  $\Psi$  that illustrate the wide array of applications that are encompassed by our framework.<sup>4</sup>

**2.2.1. Matching with Transferable Utility (TU).** The classic TU matching model has been used throughout economics—it is the cornerstone of Becker’s (1973) marriage model. TU matching with individual-specific heterogeneity been studied by Choo and Siow (2006), Galichon and Salanié (2014), and Chiappori, Salanié and Weiss (2014). TU matching is often used in modeling settings like labor markets, marriage markets, and housing markets. To recover the TU model in our framework, we take

$$\Psi_{ij}(t, t') = t + t'. \tag{2.3}$$

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<sup>4</sup>The specifications we discuss in Sections 2.2.1–2.2.4 all satisfy our main assumption on  $\Psi$  (Assumption 1, introduced in Section 3) directly; under weak conditions on the primitives of the model, the specifications in Sections 2.2.5–2.2.7 do, as well.

Hence,  $\Psi_{ij}(t_{i\leftarrow j}, t_{j\leftarrow i}) = 0$  means that  $t_{i\leftarrow j} = -t_{j\leftarrow i}$ : in algebraic terms, the amount of transfers that  $i$  receives from  $j$  is the opposite of the amount that  $j$  gets from  $i$ . In other words, there is no friction, and utility is perfectly transferable.

**2.2.2. Matching with Non-Transferable Utility (NTU).** The NTU matching model of Gale and Shapley (1962) has a history as rich as TU matching does—NTU matching is frequently used to model school choice markets, organ exchange matching, and centralized job assignment. NTU matching with individual-specific heterogeneity is more recent—it has been studied by Galichon and Hsieh (2014).

Like the TU model, we can embed the NTU model in our framework—in this case, by taking

$$\Psi_{ij}(t, t') = \max\{t, t'\}. \quad (2.4)$$

Hence,  $\Psi_{ij}(t_{i\leftarrow j}, t_{j\leftarrow i}) = 0$  implies that  $t_{i\leftarrow j} \leq 0$  and  $t_{j\leftarrow i} \leq 0$ : in other words, it is not possible to receive a positive transfer from one's partner in this model—utility is simply not transferable.

**2.2.3. Matching with Exponentially Transferable Utility (ETU).** We introduce an *Exponentially Transferable Utility (ETU)* model, in which  $\Psi_{ij}$  takes the form

$$\Psi_{ij}(t, t') = \tau_{ij} \log \left( \frac{\exp(t/\tau_{ij}) + \exp(t'/\tau_{ij})}{2} \right). \quad (2.5)$$

Here, the parameter  $\tau_{ij}$  is defined as the *degree of transferability*.<sup>5</sup> Note that  $\tau_{ij}$  is allowed to vary with  $i$  and  $j$ , so that the ease of transferring utility varies across types of couples. Using  $t_{i\leftarrow j} = u_i - \alpha_{ij}$  and  $t_{j\leftarrow i} = v_j - \gamma_{ij}$ , we obtain that feasible utilities  $u_i$  and  $v_j$  are related by

$$v_j = \gamma_{ij} + \tau_{ij} \log \left( 2 - \exp \left( \frac{u_i - \alpha_{ij}}{\tau_{ij}} \right) \right) \text{ and } u_i = \alpha_{ij} + \tau_{ij} \log \left( 2 - \exp \left( \frac{v_j - \gamma_{ij}}{\tau_{ij}} \right) \right)$$

whenever the expressions make sense.

ETU matching can be interpreted as a simple model of household consumption with logarithmic utilities: Assume that the joint budget of a household is 2, which is shared into  $c_i$  and  $c_j$ , the man and the woman's private consumptions. Assume that the man  $i$ 's

<sup>5</sup>A particular case of this model is given in Legros and Newman (2007, p. 1086).

utility is  $\alpha_{ij} + \tau_{ij} \log c_i$  and woman  $j$ 's is  $v_{ij} = \gamma_{ij} + \tau_{ij} \log c_j$ , so that agents care about the identities of their partners and about private consumption. The model then takes the form (2.5).

As  $\tau_{ij} \rightarrow 0$ , we recover the NTU model (2.4), and when  $\tau_{ij} \rightarrow +\infty$ , we obtain the TU model ( $\Psi_{ij}(t, t') \rightarrow \frac{t+t'}{2}$ , which is isomorphic to (2.3)). Hence, the ETU model interpolates between the nontransferable fully transferable utility models (see Figure 1).

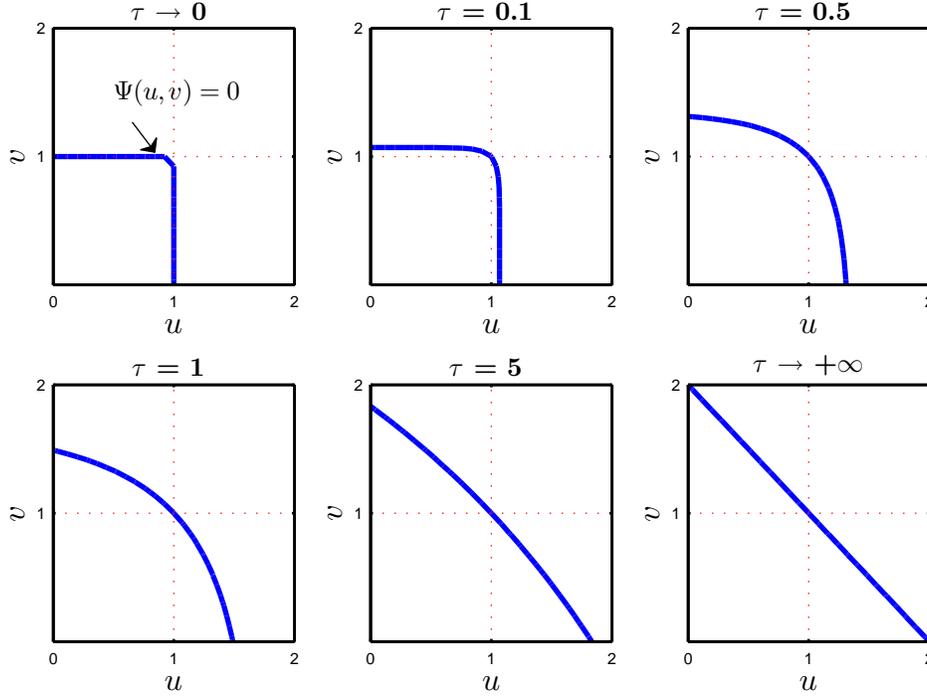


FIGURE 1. Transfer functions for the TU, NTU and ETU cases. The plots are the set of points  $(t, t')$  solution to the equation  $\tau \log((\exp(t/\tau) + \exp(t'/\tau))/2) = 0$  for a given value of  $\tau$ . A small (resp. large)  $\tau$  provides a good approximation of the NTU case (resp. TU case).

2.2.4. *Matching with a Linear Tax Schedule.* Our framework can also model a labor market with linear taxes: Assume the nominal wage  $w_{ij}$  is taxed at rate  $\theta_{ij}$  on the employee's side (income tax) and at rate  $s_{ij}$  on the firm's side (social contributions). The taxes are allowed to depend on both employer and employee characteristics.

If employee  $i$  and employer  $j$  respectively have (post-transfer) utilities  $\alpha_{ij} + (1 - \theta_{ij}) w_{ij}$  and  $\gamma_{ij} - (1 + s_{ij}) w_{ij}$ , then we have  $t_{i \leftarrow j} = (1 - \theta_{ij}) w_{ij}$  and  $t_{j \leftarrow i} = -(1 + s_{ij}) w_{ij}$ . Denoting  $\lambda_{ij} = (1 - \theta_{ij})^{-1}$ ,  $\zeta_{ij} = (1 + s_{ij})^{-1}$ , this specification is a particular case of the *Linearly Transferable Utility (LTU)* model:

$$\Psi_{ij}(t, t') = \lambda_{ij} t + \zeta_{ij} t', \quad (2.6)$$

where  $\lambda_{ij}, \zeta_{ij} > 0$ . The LTU model (2.6) extends that of Jaffe and Kominers (2014), in which  $\theta_{ij}$  could only depend on  $i$  and  $\zeta_{ij}$  could only depend on  $j$ .

**2.2.5. Matching with a Nonlinear Tax Schedule.** Our framework is general enough to extend to a nonlinear tax schedule, well beyond linear taxes. Assume that if the nominal wage is  $w_{ij}$ , the tax levied on the employee is  $\theta_{ij}(w_{ij})$  on the employee's side (income tax) and the tax levied on the firm's side is  $s_{ij}(w_{ij})$  (social contributions). The functions  $\theta_{ij}$  and  $s_{ij}$  are usually assumed convex, and they are allowed to depend on both employer and employee characteristics.

If employee  $i$  and employer  $j$  respectively have (post-transfer) utilities  $\alpha_{ij} + w_{ij} - \theta_{ij}(w_{ij})$  and  $\gamma_{ij} - w_{ij} - s_{ij}(w_{ij})$ , then we have  $t_{i \leftarrow j} = w_{ij} - \theta_{ij}(w_{ij})$  and  $t_{j \leftarrow i} = -w_{ij} - s_{ij}(w_{ij})$ . Denoting  $\lambda_{ij}(t) = (Id - \theta_{ij})^{-1}(t)$ ,  $\zeta_{ij}(t') = (Id + s_{ij})^{-1}(t')$ , this specification gives rise to the following model:

$$\Psi_{ij}(t, t') = \lambda_{ij}(t) + \zeta_{ij}(t'). \quad (2.7)$$

This model therefore allows for a nonlinear tax schedule that depends on the types of the employers and the employees.

**2.2.6. Matching with Uncertainty.** Now, we consider a model of the labor market with uncertainty regarding match quality; such as model is considered by Legros and Newman (2007) and Chade and Eeckhout (2014), who focus on characterizing positive assortativeness.

We assume that a worker of type  $i$  decides to match with a firm of type  $j$ , and decide on a wage  $w_{ij}$ . The job amenity is  $\tilde{e}_{ij}$ , where  $\tilde{e}_{ij}$  is a stochastic term learned only after the match is formed; the distribution of  $\tilde{e}_{ij}$  may depend on  $i$  and  $j$ . The employee is risk averse and has an increasing and concave utility function  $\mathfrak{U}(\cdot)$ . Then the employees' and

employers' systematic utilities are respectively  $\mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + w_{ij})]$  and  $\gamma_{ij} - w_{ij}$ . This model can be recast as a matching model with imperfectly transferable utility by noting that  $t_{i \leftarrow j} = \mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + w_{ij})]$ ,  $t_{j \leftarrow i} = -w_{ij}$ , so that

$$\Psi_{ij}(t, t') = t - \mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + \gamma_{ij} - t')].$$

*2.2.7. Collective Models with Spillovers and Public Goods.* We consider a situation in which a man  $i$  and a woman  $j$  have respective utilities  $u(c, l, g; i)$  and  $v(c', l', g; j)$  over private consumption  $c$  and  $c'$ , private leisure  $l$  and  $l'$ , and a public good  $g$ . The wages of a man of type  $i$  and of a woman of type  $j$  are respectively denoted  $w_i$  and  $w_j$ , and the price of the public good is denoted  $p$ . The budget constraint of the household is  $c + c' + w_i l + w_j l' + pg = \Phi_{ij}$ , where  $\Phi_{ij}$  is the total combined potential income of the pair.<sup>6</sup>

We define  $\mathfrak{U}_i(t, g) = \max\{u(c, l, g; i) : c + lw_i = t\}$  and  $\mathfrak{V}_j(t', g) = \max\{v(c', l', g; j) : c + lw_i = t'\}$ , the indirect utilities of the man and the woman if the private part of the household budget  $\Phi_{ij} - pg = t + t'$  is split into  $t$  for the man and  $t'$  for the woman. The “collective” approach initiated by Chiappori (1988) assumes that the outcome  $(u, v)$  lies on the Pareto frontier of the feasible set of achievable utilities, given some sharing rule. Letting  $\theta$  and  $1 - \theta$  be the respective Pareto weights associated to the man and the woman's utility, the household's problem becomes

$$\max_{t, t', g} \{\theta \mathfrak{U}_i(t, g) + (1 - \theta) \mathfrak{V}_j(t', g) : t + t' + pg = \Phi_{ij}\}.$$

The first-order condition implies that  $\partial_g \mathfrak{U}_i / \partial_t \mathfrak{U}_i + \partial_g \mathfrak{V}_j / \partial_{t'} \mathfrak{V}_j = p$ , which implicitly defines  $g$  as a function of  $u = \mathfrak{U}_i(t, g)$  and  $v = \mathfrak{V}_j(t', g)$ . Letting  $\mathfrak{U}_i^{-1}(\cdot, g)$  and  $\mathfrak{V}_j^{-1}(\cdot, g)$  be the inverses of  $\mathfrak{U}_i(\cdot, g)$  and  $\mathfrak{V}_j(\cdot, g)$ , respectively, the collective model can be reformulated in our framework by taking

$$\Psi_{ij}(u, v) = \mathfrak{U}_i^{-1}(u, G(u, v)) + \mathfrak{V}_j^{-1}(v, G(u, v)) + pG(u, v) - \Phi_{ij};$$

in the case without a public good, the model takes the simpler form

$$\Psi_{ij}(u, v) = \mathfrak{U}_i^{-1}(u) + \mathfrak{V}_j^{-1}(v) - \Phi_{ij}.$$

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<sup>6</sup>Because of heterogeneity in spillovers, economies of scale, and similar, we do not assume that  $\Phi_{xy}$  is separatively additive.

The classical collective approach assumes that the outcome  $(u, v)$  lies on the Pareto frontier of the feasible set of achievable utilities, given some exogenous sharing rule. In the present context, the sharing rule is determined endogenously through matching model. An empirical matching framework for the collective model is given in Choo and Seitz (2013).

2.2.8. *Matching with investments.* Following Noldeke and Samuelson (2014), who make an explicit link with ITU matching, one should distinguish between ex-ante equilibrium and ex-post equilibrium in the matching problem with investments.

In an *ex-post equilibrium* with investments, the matching occurs after investments have been made; investment decisions are made rationally in a first stage in anticipation of the outcome of the matching game taking place at the second stage. The matching problem is then a standard matching problem that takes investment as exogenous.

In contrast, in an *ex-ante equilibrium* with investments, investments are decided simultaneously with matching, and agents commit with their investments and the sharing of the surplus. Hence, if  $a$  is the investment decision of the man and  $b$  is the investment decision of the woman, the feasible frontier should depend on the investment decisions of the man and the woman; hence, conditional on investments  $a$  and  $b$ , the set of utilities  $(u, v)$  attainable are given by  $\Psi_{ij}(u, v; a, b) \leq 0$ . However, it is possible to retain only the efficient outcomes, namely, consider

$$\Psi_{i,j}(u, v) = \max_{a,b} \{\Psi_{i,j}(u, v; a, b)\}$$

and hence, the problem can be seen as belonging to the ITU setting. Note that even if we started with a TU model, namely if  $\Psi_{ij}(u, v; a, b) = u + v - \Phi_{ij}(a, b)$ , then the ex-ante equilibrium would no longer be a TU problem, as  $\Psi_{i,j}(u, v) = \max_{a,b} \{u + v - \Phi_{ij}(a, b)\}$ , which is genuinely an ITU matching problem. Again, we refer to Noldeke and Samuelson (2014) for much more on these problems.

### 3. UNOBSERVED HETEROGENEITY

We now assume that individuals may be gathered in groups of agents of similar observable characteristics, or types, but heterogeneous tastes. We let  $\mathcal{X}$  and  $\mathcal{Y}$  be the sets of *types* of men and women, respectively; we assume that  $\mathcal{X}$  and  $\mathcal{Y}$  are finite. Let  $x_i \in \mathcal{X}$  (resp.  $y_j \in \mathcal{Y}$ )

be the type of individual man  $i$  (resp. woman  $j$ ). We let  $n_x$  be the number of men of type  $x$ , and let  $m_y$  be the number of women of type  $y$ . In the sequel, we denote by  $\mathcal{X}_0 \equiv \mathcal{X} \cup \{0\}$  the set of marital options available to women (either type of male partner or singlehood, denoted 0); analogously,  $\mathcal{Y}_0 = \mathcal{Y} \cup \{0\}$  denotes the set of marital options available to men (either type of female partner or singlehood, again denoted 0). We assume in the sequel that  $\Psi_{ij}(\cdot, \cdot)$  depends only on agent types—that is,  $\Psi_{ij}(\cdot, \cdot) \equiv \Psi_{x_i y_j}(\cdot, \cdot)$ .

If man  $i$  and woman  $j$  match, then (before bargaining) they enjoy pre-transfer *match affinities*  $\alpha_{x_i y_j} + \varepsilon_{iy}$  and  $\gamma_{x_i y_j} + \eta_{xj}$  respectively. If man  $i$  (resp. woman  $j$ ) decides to remain single, he (resp. she) receives  $\varepsilon_{i0}$  (resp.  $\eta_{0j}$ ). Here,  $(\varepsilon_{iy})_{y \in \mathcal{Y}_0}$  and  $(\eta_{xj})_{x \in \mathcal{X}_0}$  are the idiosyncratic, random parts of the agents' affinities—which we assume are i.i.d. random vectors respectively valued in  $\mathbb{R}^{\mathcal{Y}_0}$  and  $\mathbb{R}^{\mathcal{X}_0}$ . We assume that  $\varepsilon_{iy}$  and  $\eta_{jx}$  are respectively distributed according to  $\mathbf{P}_x$  and  $\mathbf{Q}_y$ .

Agents  $i$  and  $j$  who match together may decide on a *transfer*  $(t_{i \leftarrow j}, t_{j \leftarrow i})$  that specifies the amount  $t_{i \leftarrow j}$  transferred from  $j$  to  $i$ , along with the amount  $t_{j \leftarrow i}$  transferred from  $i$  to  $j$ . Any such transfer must satisfy the feasibility condition (2.1)—that is, we must have

$$\Psi_{x_i y_j}(t_{i \leftarrow j}, t_{j \leftarrow i}) \leq 0.$$

After the transfer,  $i$ 's and  $j$ 's utilities are respectively given by

$$u_i = \alpha_{x_i y_j} + t_{i \leftarrow j} + \varepsilon_{iy} \quad \text{and}$$

$$v_j = \gamma_{x_i y_j} + t_{j \leftarrow i} + \eta_{xj}.$$

In other words, when man  $i$  and woman  $j$  bargain, they may reach any utility levels  $u_i$  and  $v_j$  that satisfy (2.1), where  $\Psi_{ij} = \Psi_{x_i y_j}$ ,  $\alpha_{ij} = \alpha_{x_i y_j} + \varepsilon_{iy}$ , and  $\gamma_{ij} = \gamma_{x_i y_j} + \eta_{xj}$ .

Our principal restriction here is that *the transfer function only depends on the observable types of both partners*. This restriction extends the “additive separability assumption” highlighted by Chiappori, Salanié and Weiss (2014), building on the work of Choo and Siow (2006). In the case of TU models (see Example 2.2.1 below), our restriction simply states that the surplus  $\Phi_{ij}$  can be decomposed in the form  $\Phi_{ij} = \Phi_{x_i y_j} + \varepsilon_{iy} + \eta_{jx}$ . Note that, while the transfers  $t_{i \leftarrow j}$  and  $t_{j \leftarrow i}$  are allowed to vary idiosyncratically, it will be a property

of the equilibrium (stated in part (iii) of Theorem 1) that they will actually only depend on agent types, so that  $t_{i \leftarrow j} = t_{x_i \leftarrow y_j}$  and  $t_{j \leftarrow i} = t_{y_j \leftarrow x_i}$ .

**Assumption 1.** *We assume:*

- (a) *For any  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ , we have  $\Psi_{xy}(\cdot, \cdot)$  continuous.*
- (b) *For any  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ ,  $t \leq t'$  and  $r \leq r'$ , we have  $\Psi_{xy}(t, r) \leq \Psi_{xy}(t', r')$ ; furthermore, when  $t < t'$  and  $r < r'$ , we have  $\Psi_{xy}(t, r) < \Psi_{xy}(t', r')$ .*
- (c) *For any sequence  $(t_n, r_n)$ , if  $(r_n)$  is bounded and  $t_n \rightarrow +\infty$ , then  $\liminf \Psi_{xy}(t_n, r_n) > 0$ . Analogously, if  $(t_n)$  is bounded and  $r_n \rightarrow +\infty$ , then  $\liminf \Psi_{xy}(t_n, r_n) > 0$ .*
- (d) *For any sequence  $(t_n, r_n)$  such that if  $(t_n - r_n)$  is bounded and  $t_n \rightarrow -\infty$  (or equivalently,  $r_n \rightarrow +\infty$ ), we have that  $\limsup \Psi_{xy}(t_n, r_n) < 0$ .*

All the components of Assumption 1 are extremely natural: 1 (a) and 1 (b) respectively express continuity and monotonicity of the transfer function. Assumption 1 (c) requires that it is not feasible to provide an arbitrarily high amount of utility to an individual while keeping the utility of his or her partner constant. Assumption 1 (d) requires that there is always a feasible transfer solution that offers a bounded utility difference between partners.

We also impose a simple restriction on the heterogeneity distributions  $\mathbf{P}_x$  and  $\mathbf{Q}_y$ .

**Assumption 2.**  *$\mathbf{P}_x$  and  $\mathbf{Q}_y$  have full support and are absolutely continuous with respect to the Lebesgue measure.*

Assumption 2 implies in particular that for any pair  $x$  and  $y$  of men's and women's types, there will be a man  $i$  of type  $x$  and a woman  $j$  of type  $y$  such that man  $i$  prefers type  $y$  to any other women's type, and such that woman  $j$  prefers type  $x$  to any other men's type. This introduces a relatively strong restriction on the equilibrium matching: every possible matching of types will be observed in equilibrium.

Following Galichon and Salanié (2014), we introduce the discrete choice problem of a man  $i$  of type  $x$  facing utility  $U_{xy} + \varepsilon_{iy}$  from matching with a partner of type  $y$ , and of a woman  $j$  of type  $y$  facing utility  $V_{xy} + \eta_{jx}$  of matching with type  $x$ . We define the total

indirect surplus of men and women by

$$G(U) = \sum_x n_x \mathbb{E} \left[ \max_y \{U_{x,y} + \varepsilon_{iy}, \varepsilon_{i0}\} \right] \quad \text{and} \quad H(V) = \sum_y m_y \mathbb{E} \left[ \max_x \{V_{xy} + \eta_{jx}, \eta_{j0}\} \right]. \quad (3.1)$$

The number of men of type  $x$  choosing a partner of type  $y$  is a number  $\mu_{xy} = \partial G(U) / \partial U_{xy}$ , which we denote in vector notation by  $\mu \equiv \nabla G(U)$ . In general, this vector  $\mu$  is a *semi-matching (of men)* in the sense that it satisfies  $\sum_y \mu_{xy} \leq n_x$  for each  $x$ , but does not necessarily satisfy the other set of constraints. (That is, each man chooses at most one woman, but the same woman may be chosen by several different men.) Similarly, the number of women of type  $y$  choosing a partner of type  $x$  is given by  $\nu \equiv \nabla H(V)$ . This is a *semi-matching (of women)* in the sense that it satisfies  $\sum_x \nu_{xy} \leq m_y$  for each  $y$ , but does not necessarily satisfy the other set of constraints.

We need to invert  $\mu = \nabla G(U)$  and  $\nu = \nabla H(V)$  in order to express  $U$  and  $V$  as a function of  $\mu$  and  $\nu$  respectively. To do this, we introduce the Legendre-Fenchel transform (a.k.a. convex conjugate) of  $G$  and  $H$ :

$$G^*(\mu) = \sup_U \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\} \quad \text{and} \quad H^*(\nu) = \sup_V \left\{ \sum_{xy} \nu_{xy} V_{xy} - H(V) \right\}. \quad (3.2)$$

It is a well-known fact from convex analysis (cf. Rockafellar 1970) that  $\mu = \nabla G(U)$  if and only if  $U = \nabla G^*(\mu)$ , while  $\nu = \nabla H(V)$  if and only if  $V = \nabla H^*(\nu)$ . Galichon and Salanié (2014) go beyond McFadden's Generalized Extreme Value setting, discussing thoroughly various choices for distributions  $\mathbf{P}_x$  and  $\mathbf{Q}_y$ , such as the Random Uniform Scalar Coefficient model. The simplest example is the logit specification, as was used in the original Choo and Siow (2003) framework.

In Section 4, we consider the particular case where  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  are the distribution of independent Gumbel random variables, in order to be in the logit framework, which will be labeled as Assumption 2'.

Finally, we will prove uniqueness of an equilibrium in the case when  $\Psi_{xy}$  is differentiable, that is when the marginal rate of substitution of partner's utilities are finite.

**Assumption 3.** For each  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ ,  $(t, t') \rightarrow \Psi_{xy}(t, t')$  is differentiable.

While satisfied in most of examples of interest, this assumption will not be met for the NTU case. In this case, it can be shown that aggregate equilibrium matching is not unique in general.<sup>7</sup>

**3.1. Equilibrium Characterization.** An *aggregate matching* (or just a *matching*, when the abuse of terminology will not introduce confusion), is specified by a vector  $(\mu_{xy})_{x \in \mathcal{X}_0, y \in \mathcal{Y}_0}$  counting the numbers of matches between men of type  $x$  and women of type  $y$ . Let  $\mathcal{M}$  be the set of matchings, that is, the set of  $\mu_{xy} \geq 0$  such that  $\sum_{y \in \mathcal{Y}} \mu_{xy} \leq n_x$  and  $\sum_{x \in \mathcal{X}} \mu_{xy} \leq m_y$ . For later purposes, we shall need to consider the strict interior of  $\mathcal{M}$ , denoted  $\mathcal{M}^{\text{int}}$ , i.e. the set of  $\mu_{xy} > 0$  such that  $\sum_{y \in \mathcal{Y}} \mu_{xy} < n_x$  and  $\sum_{x \in \mathcal{X}} \mu_{xy} < m_y$ . The elements of  $\mathcal{M}^{\text{int}}$  are called *interior matchings*.

As we noted in Section 2.1, stability implies that  $\Psi_{ij}(u_i - \alpha_{ij}, v_j - \gamma_{ij}) \geq 0$ , with equality if  $i$  and  $j$  are matched. Using Assumption 1, we can re-express this as

$$\Psi_{x_i y_j} \left( u_i - \alpha_{x_i y_j} - \varepsilon_{iy}, v_j - \gamma_{x_i y_j} - \eta_{jx} \right) \geq 0,$$

with the same equality condition. Hence, for all pairs  $x$  and  $y$ , we have the inequality

$$\min_{i: x_i=x} \min_{j: y_j=y} \left\{ \Psi_{x_i y_j} \left( u_i - \varepsilon_{iy} - \alpha_{xy}, v_j - \eta_{jx} - \gamma_{xy} \right) \right\} \geq 0,$$

with equality if  $\mu_{xy} > 0$ , that is, if there is at least one marriage between a man of type  $x$  and a woman of type  $y$ . Taking

$$U_{xy} = \min_{i: x_i=x} \{u_i - \varepsilon_{iy}\} \quad \text{and} \quad V_{xy} = \min_{j: y_j=y} \{v_j - \eta_{jx}\}, \quad (3.3)$$

thus, making use of the monotonicity of  $\Psi$  in Assumption 1, matching  $\mu \in \mathcal{M}$  is an equilibrium matching if inequality  $\Psi_{xy}(U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}) \geq 0$  holds for any  $x$  and  $y$ , with equality if  $\mu_{xy} > 0$ .

**Definition 1.** The triple  $(\mu_{xy}, U_{xy}, V_{xy})_{x \in \mathcal{X}_0, y \in \mathcal{Y}_0}$  is an *Equilibrium Matching with Random Utility (EMRU)* if the following three conditions are met:

- (i)  $\mu$  is an interior matching, i.e.  $\mu \in \mathcal{M}^{\text{int}}$ ;

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<sup>7</sup>We are grateful to Yu-Wei Hsieh for pointing out to us a crucial example of nonuniqueness in the NTU case.

(ii)  $(U, V)$  is feasible, i.e.

$$\Psi_{xy}(U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}) = 0; \quad (3.4)$$

(iii)  $\mu$ ,  $U$ , and  $V$  are related by market clearing condition

$$\mu = \nabla G(U) = \nabla H(V). \quad (3.5)$$

As we remarked, Assumption 2 ensures that (with probability 1) there will be a man  $i$  of type  $x$  and a woman  $j$  of type  $y$  such that  $i$  prefers type  $y$  and  $j$  prefers type  $x$ . Thus, under Assumption 2, at any (aggregate) equilibrium matching, we have  $\mu_{xy} > 0$  for all  $x$  and  $y$ ; that is,  $\mu \in \mathcal{M}^{\text{int}}$ .

A simple count of variables shows that  $(\mu, U, V)$  is of dimension  $3 \times |\mathcal{X}| \times |\mathcal{Y}|$ . This number coincides with the number of equations provided by (3.4) and (3.5). However, this observation just provides a sanity check—it does not directly imply existence or uniqueness of equilibrium.

The following theorem shows that any equilibrium matching boils down to a simple equation on  $\mu$ , from which all other quantities of interest in the problem can be deduced.

**Theorem 1.** *Under Assumptions 1 and 2:*

(i) *A matching  $\mu \in \mathcal{M}^{\text{int}}$  is an equilibrium matching if and only if it solves the fundamental matching equation*

$$\Psi(\nabla G^*(\mu) - \alpha, \nabla H^*(\mu) - \gamma) = 0. \quad (3.6)$$

(ii) *If  $\mu \in \mathcal{M}^{\text{int}}$  is an equilibrium matching, then the associated systematic utilities  $U_{xy}$  and  $V_{xy}$  are given by*

$$U = \nabla G^*(\mu) \text{ and } V = \nabla H^*(\mu). \quad (3.7)$$

(iii) *If  $(\mu, U, V)$  is an equilibrium outcome, then individual equilibrium utilities  $u_i$  and  $v_j$  are given by*

$$u_i = \max \left\{ \max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}\}, \varepsilon_{i0} \right\} \text{ and } v_j = \max \left\{ \max_{x \in \mathcal{X}} \{V_{xy} + \eta_{jx}\}, \eta_{j0} \right\}. \quad (3.8)$$

(iv) The equilibrium transfers  $t_{i \leftarrow j}$  and  $t_{j \leftarrow i}$  are given by  $t_{i \leftarrow j} = t_{x_i \leftarrow y_j}$  and  $t_{j \leftarrow i} = t_{y_j \leftarrow x_i}$ , where

$$t_{x \leftarrow y} = U_{xy} - \alpha_{xy} \text{ and } t_{y \leftarrow x} = V_{xy} - \gamma_{xy}. \quad (3.9)$$

In particular, the transfers between  $i$  and  $j$  only depend on the observable types  $x_i$  and  $y_j$ .

As implied by part (i) of this result, the fundamental equilibrium equation is (3.6), which is solely an equation in  $\mu$ . Once this equation is solved, the systematic utilities  $U_{xy}$  and  $V_{xy}$  are deduced in part (ii) by Equations (3.7), from which the individual utilities  $u_i$  and  $v_j$  are deduced in part (iii) by (3.8). The equilibrium transfers are in turn deduced from (3.9).

**3.2. Remarks.** In the TU setting,  $\Psi_{xy}(t, t') = t + t'$ ; thus, the fundamental matching equation (3.6) can be rewritten as

$$\nabla G^*(\mu) + \nabla H^*(\mu) = \alpha + \gamma.$$

For this specification of  $\Psi$ , and for fully general  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  satisfying Assumption 2, Galichon and Salanié (2014) have shown the existence and uniqueness of a solution to (3.6), by showing that this equation coincides with the first-order conditions associated to the utilitarian welfare maximization problem, namely

$$\max_{\mu} \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \mathcal{E}(\mu) \right\}$$

where  $\Phi = \alpha + \gamma$  is the systematic part of the joint affinity, and  $\mathcal{E} := G^* + H^*$  is an entropy penalization that trades-off against the maximization of the observable part of the joint affinity.

Note that the equilibrium defined in Equation 3.6 belongs to the class of Nonlinear Complementarity Problems (NCP): indeed, letting

$$f(\mu) = \Psi(\nabla G^*(\mu) - \alpha, \nabla H^*(\mu) - \gamma),$$

one looks for  $\mu \geq 0$  such that  $f(\mu) \geq 0$  and  $\mu \cdot f(\mu) = 0$ .

In the NTU setting, i.e. when  $\Psi_{xy}(t, t') = \max(t, t')$ , (3.6) takes the form

$$\max \left( \frac{\partial G^*(\mu)}{\partial \mu_{xy}} - \alpha_{xy}, \frac{\partial H^*(\mu)}{\partial \mu_{xy}} - \gamma_{xy} \right) = 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

For this specification of  $\Psi$ , and for fully general  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  satisfying Assumption 2, Galichon and Hsieh (2014) show existence and computation of a solution in full generality via an aggregate version of the Gale–Shapley (1962) algorithm. In general, however, the problem of existence and uniqueness of a stable matching solution to (3.6) (with general  $\Psi$  satisfying Assumption 1 and general  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  satisfying Assumption 2) remains an open problem. In Section 4, we prove existence and uniqueness of a solution to (3.6) with general  $\Psi$  satisfying Assumption 1, but in the particular case that the heterogeneity structure is logit.

Part (ii) of Theorem 1 simply expresses that  $U$ ,  $V$  and  $\mu$  are related by  $U = \nabla G^*(\mu)$  and  $V = \nabla H^*(\mu)$ . By (3.6),  $U$  and  $V$  thus defined are automatically feasible.

Using (3.3) enables us to construct  $(U_{xy})$  and  $(V_{xy})$  as a function of  $(u_i)$  and  $(v_j)$ . Part (iii) of Theorem 1 provides a converse:  $(u_i)$  and  $(v_j)$  can be determined from  $(U_{xy})$  and  $(V_{xy})$ , by way of equation (3.8). Consequently, we see that at equilibrium, each man in the market solves a discrete choice problem with systematic utility  $U_{xy}$ , and each woman solves a discrete choice problem with systematic utility  $V_{xy}$ .

Finally, part (iv) of Theorem 1 implies that agents keep their entire utility shocks at equilibrium, even when they could transfer them fully or partially. This finding, which carries strong testable implications, was known in the TU case (see Chiappori, Salanié and Weiss (2014)). Our theorem clarifies the deep mechanism that drives this result: the crucial assumption is that the transfer function  $\Psi_{ij}$  should only depend on  $i$  and  $j$  through the observable types  $x_i$  and  $y_j$ .

#### 4. THE IMPERFECTLY TRANSFERABLE UTILITY MODEL WITH LOGIT HETEROGENEITY (ITU-LOGIT)

For the remainder of the paper, we specialize Assumption 2 to the logit case.

**Assumption 2’.**  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  are the distributions of *i.i.d.* standard type I extreme value random variables.

**4.1. Aggregate Matching Functions.** In the logit case, it is well-known that  $G$  and  $G^*$  above can be expressed in closed-form by

$$G(U) = \sum_x n_x \log \left( 1 + \sum_{y \in \mathcal{Y}} \exp(U_{xy}) \right) \quad \text{and} \quad G^*(\mu) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}_0} \mu_{xy} \log \frac{\mu_{xy}}{n_x}.$$

where  $\mu_{x0} = n_x - \sum_{y \in \mathcal{Y}} \mu_{xy}$ . The relations between  $U$  and  $\mu$  are

$$U_{xy} = \log \left( \frac{\mu_{xy}}{\mu_{x0}} \right) \quad \text{and} \quad \mu_{xy} = \mu_{x0} \exp(U_{xy}). \quad (4.1)$$

Now, (4.1) yields  $U_{xy} = \log \mu_{xy} - \log \mu_{x0}$  and  $V_{xy} = \log \mu_{xy} - \log \mu_{0y}$ . Hence, the equilibrium equation (3.6) becomes

$$\Psi_{xy} (\log \mu_{xy} - \log \mu_{x0} - \alpha_{xy}, \log \mu_{xy} - \log \mu_{0y} - \gamma_{xy}) = 0,$$

which implicitly defines  $\mu_{xy}$  as a function of  $\mu_{x0}$  and  $\mu_{0y}$ :

$$\mu_{xy} = M_{xy} (\mu_{x0}, \mu_{0y}). \quad (4.2)$$

**Definition 2.** Maps  $(\mu_{x0}, \mu_{0y}) \mapsto M_{xy} (\mu_{x0}, \mu_{0y})$  are called *aggregate matching functions (AMF)*.

The aggregate matching function concept is not new (see Petrongolo and Pissarides (2001) and Siow (2008)), but there is an important difference between our aggregate matching function and much of the prior work. Here,  $\mu_{x0}$  and  $\mu_{0y}$  are the number of men and women selected into singlehood, which is endogenous, and determined by equilibrium equations (4.7). In the demography literature (up to the important exception of Choo and Siow (2006) and the subsequent literature),  $\mu_{x0}$  and  $\mu_{0y}$  are usually the number of *available* men and women, assumed to be exogenous.

Assumption 1 implies a number of properties of  $M_{xy}$  (see Lemma 1 in Appendix B). In particular, the map  $(a, b) \mapsto M_{xy} (a, b)$  is continuous and weakly isotone.

**4.2. Example Specifications, Revisited.**

4.2.1. *TU-logit Specification.* In the logit case of the TU specification introduced in Section 2.2.1, the matching function becomes

$$\mu_{xy} = \mu_{x0}^{1/2} \mu_{0y}^{1/2} \exp \frac{\Phi_{xy}}{2}, \quad (4.3)$$

which is Choo and Siow's (2006) formula.

4.2.2. *NTU-logit Specification.* In the logit case of the NTU specification introduced in Section 2.2.2, the matching function becomes

$$\mu_{xy} = \min (\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}). \quad (4.4)$$

When  $\mu_{x0} e^{\alpha_{xy}} \leq \mu_{0y} e^{\gamma_{xy}}$ ,  $\mu_{xy} = \mu_{x0} e^{\alpha_{xy}}$  is constrained by the choice problem of men; we say that, relative to pair  $xy$ , men are on the *short side (of the market)* and women are on the *long side (of the market)*, and visa versa. In Section 5, we see that whether one is on the long or short side affects one's welfare (and welfare comparative statics).

Note that Dagsvik (2000) and Menzel (2014) obtain  $\mu_{xy} = \mu_{x0} \mu_{0y} e^{\alpha_{xy} + \gamma_{xy}}$ —different from our formula (4.4). The reason for this difference is that Dagsvik (2000) and Menzel (2014) assume that the stochastic matching affinities are given by  $\alpha_{ij} = \alpha_{xy} + \varepsilon_{ij}$  and  $\gamma_{ij} = \gamma_{xy} + \eta_{ij}$ , where the  $\varepsilon_{ij}$  and  $\eta_{ij}$  terms are i.i.d. type I extreme value distributions. In contrast, in our setting,  $\alpha_{ij} = \alpha_{xy} + \varepsilon_{ij}$  and  $\gamma_{ij} = \gamma_{xy} + \eta_{xij}$ .

4.2.3. *ETU-logit Specification.* In the logit case of the the Exponentially Transferable Utility specification introduced in Section 2.2.3, the feasibility frontier takes the form

$$\exp \left( \frac{U_{xy} - \alpha_{xy}}{\tau_{xy}} \right) + \exp \left( \frac{V_{xy} - \gamma_{xy}}{\tau_{xy}} \right) = 2,$$

which, when combined with identification formulae  $\exp U_{xy} = \mu_{xy} / \mu_{x0}$  and  $\exp V_{xy} = \mu_{xy} / \mu_{0y}$ , yields the following expression for the matching function:

$$\mu_{xy} = \left( \frac{e^{-\alpha_{xy}/\tau_{xy}} \mu_{x0}^{-1/\tau_{xy}} + e^{-\gamma_{xy}/\tau_{xy}} \mu_{0y}^{-1/\tau_{xy}}}{2} \right)^{-\tau_{xy}}. \quad (4.5)$$

As expected, when  $\tau_{xy} \rightarrow 0$ , formula (4.5) converges to the NTU-logit formula, (4.4). Likewise, when  $\tau_{xy} \rightarrow +\infty$ , (4.5) converges to the TU-logit formula, (4.3).

But when  $\tau_{xy} = 1$ , then (up to multiplicative constants)  $\mu_{xy}$  becomes the harmonic mean between  $\mu_{x0}$  and  $\mu_{0y}$ . We thus recover a classical matching function form—the “Harmonic Marriage Matching Function” that has been used by demographers for decades (see, e.g., Qian and Preston (1993) and Schoen (1981)).

To our knowledge, our framework gives the first behavioral justification of harmonic marriage matching function. Indeed, as Siow (2008, p. 5) argued, this choice of matching function heretofore had “no coherent behavioral foundation.”

4.2.4. *LTU-logit Specification.* In the logit case of the the Linearly Transferable Utility specification introduced in Section 2.2.4, the matching function becomes

$$\mu_{xy} = e^{(\lambda_{xy}\alpha_{xy} + \zeta_{xy}\gamma_{xy})/(\lambda_{xy} + \zeta_{xy})} \mu_{x0}^{\lambda_{xy}/(\lambda_{xy} + \zeta_{xy})} \mu_{0y}^{\zeta_{xy}/(\lambda_{xy} + \zeta_{xy})}. \quad (4.6)$$

In particular, when  $\lambda_{xy} = 1$  and  $\zeta_{xy} = 1$ , we again recover the Choo and Siow (2006) identification formula.

4.3. **Equilibrium.** Combining relation (4.2) with the feasibility equations expressing  $\mu \in \mathcal{M}$  yields the following equilibrium equations for the ITU-logit model:

$$\begin{cases} (\sum_y M_{xy}(\mu_{x0}, \mu_{0y})) + \mu_{x0} = n_x \\ (\sum_x M_{xy}(\mu_{x0}, \mu_{0y})) + \mu_{0y} = m_y, \end{cases} \quad (4.7)$$

a system of  $|\mathcal{X}| + |\mathcal{Y}|$  equations in the same number of unknowns.

The following result provides the existence of an equilibrium matching in the ITU-logit case.

**Theorem 2.** *Under Assumptions 1 and 2’:*

- (i) *An equilibrium matching  $\mu \in \mathcal{M}$  exists.*
- (ii) *At the equilibrium matching  $\mu$ , the surplus levels  $u_i$  and  $v_j$  are given by (3.8), that is*

$$u_i = \max \left\{ \max_y \{U_{x_i y} + \varepsilon_{i y}\}, \varepsilon_{i 0} \right\} \text{ and } v_j = \max \left\{ \max_x \{V_{x y_j} + \eta_{j x}\}, \eta_{j 0} \right\},$$

where

- (1) •  $U_{xy} = \log(\mu_{xy}/\mu_{x0})$ ,
- $V_{xy} = \log(\mu_{xy}/\mu_{0y})$ ,

- $\mu_{x0} = n_x - \sum_y \mu_{xy}$ , and
- and  $\mu_{0y} = m_y - \sum_x \mu_{xy}$ .

The proof of Theorem 2 is constructive and shows that the equilibrium matching  $\mu$  can be obtained as the limit of the following iterative procedure.

**Algorithm 1.**

Step 0	Fix the initial value of $\mu_{0y}$ at $\mu_{0y}^0 = m_y$ .
Step $2t + 1$	Keep the values $\mu_{0y}^{2t}$ fixed. For each $x \in \mathcal{X}$ , solve for the value, $\mu_{x0}^{2t+1}$ , of $\mu_{x0}$ such that equality $\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x$ holds.
Step $2t + 2$	Keep the values $\mu_{x0}^{2t+1}$ fixed. For each $y \in \mathcal{Y}$ , solve for which is the value, $\mu_{0y}^{2t+2}$ , of $\mu_{0y}$ such that equality $\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y$ holds.

The algorithm terminates when  $\sup_y |\mu_{0y}^{2t+2} - \mu_{0y}^{2t}| < \epsilon$ .

The proof of Theorem 2 implies that Algorithm 1 converges to an equilibrium matching as  $\epsilon \rightarrow 0$ .

We now provide a result guaranteeing uniqueness, under a supplementary smoothness assumption on  $\Psi$ .

**Theorem 3.** *Under Assumptions 1, 2', and 3, the equilibrium matching  $\mu \in \mathcal{M}$  is unique.*

**4.4. Identification.** In this section, we focus on the case where

$$\Psi_{xy}(t, t') = \psi_{xy} \left( \frac{t}{\tau_{xy}}, \frac{t'}{\tau_{xy}} \right) \quad (4.8)$$

and  $\psi_{xy}$  is a known function. The parameters to be identified are the match affinities of men,  $\alpha_{xy}$ , the match affinities of women,  $\gamma_{xy}$ , and the transferability parameter,  $\tau_{xy}$ . This parameter space is of dimension  $3 \times |\mathcal{X}| \times |\mathcal{Y}|$ , so we have no hope of achieving point identification on the basis of just the matching  $\mu$  in a single market.

Indeed, we have three times more unknowns ( $\alpha_{xy}$ ,  $\gamma_{xy}$ , and  $\tau_{xy}$ ) than observed outcome parameters ( $\mu_{xy}$ ). Thus, we need at least three markets to achieve identification.<sup>8</sup> Let  $N^{\text{mkts}} \in \mathbb{N}$  be the number of markets, and let  $\mu^k$  be the matching in market  $k \in \{1, \dots, N^{\text{mkts}}\}$ .

<sup>8</sup>Of course, we also need to assume that the parameters  $\alpha_{xy}$ ,  $\gamma_{xy}$ , and  $\tau_{xy}$  are fixed across markets.

**Proposition 1.** *The identified set is the set of vectors  $(\alpha, \gamma, \tau)$  such that for all  $k \in \{1, \dots, N^{\text{mkts}}\}$ , and for all  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ , the following equalities hold:*

$$\psi_{xy} \left( \frac{\ln \mu_{xy}^k - \ln \mu_{x0}^k - \alpha_{xy}}{\tau_{xy}}, \frac{\ln \mu_{xy}^k - \ln \mu_{0y}^k - \gamma_{xy}}{\tau_{xy}} \right) = 0. \quad (4.9)$$

Note that even with a single market, the identified set described in Proposition 1 can be empty. Indeed, if  $\mu_{xy} = 0$  for some  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ , then  $\mu \notin \mathcal{M}^{\text{int}}$ , so that  $\mu$  cannot be a market equilibrium. On the other hand, even with a large number of markets, there may be multiple elements in the identified set. (E.g., when  $\psi(t, t') = t + t'$ , only the sum  $\alpha_{xy} + \gamma_{xy}$  is identified.)

In the ETU-logit case,  $\alpha_{xy}$ ,  $\gamma_{xy}$  and  $\tau_{xy}$  satisfy the relation

$$e^{-\alpha_{xy}/\tau_{xy}} \left( \mu_{x0}^k \right)^{-1/\tau_{xy}} + e^{-\gamma_{xy}/\tau_{xy}} \left( \mu_{0y}^k \right)^{-1/\tau_{xy}} = 2 \left( \mu_{xy}^k \right)^{-1/\tau_{xy}},$$

which provides a strategy for identifying the parameters  $\alpha_{xy}$ ,  $\gamma_{xy}$ , and  $\tau_{xy}$  with only three markets.

When only a single market is observed, we need to incorporate restrictions on  $\alpha$ ,  $\gamma$  and  $\tau$  in order to restore point identification. One possible route is to fix  $\tau_{xy}$  and impose the restriction  $\alpha_{xy} = \gamma_{xy}$ , in which case the previous formulas provide following identification formulae for  $\alpha$  in the TU, NTU, and ETU cases, respectively:

- In the TU case,  $\exp(\alpha_{xy}) = \mu_{xy} (\mu_{x0} \mu_{0y})^{-1/2}$ ,
- In the NTU case,  $\exp(\alpha_{xy}) = \mu_{xy} \min(\mu_{x0}, \mu_{0y})^{-1}$ ,
- In the ETU case,  $\exp(\alpha_{xy}) = \mu_{xy} \left( (\mu_{x0}^{-1/\tau_{xy}} + \mu_{0y}^{-1/\tau_{xy}}) / 2 \right)^{\tau_{xy}}$  for a fixed value of  $\tau_{xy}$ .

Last, but not least, it seems reasonable to adopt a lower-dimensional parametrization of  $\alpha_{xy}$ ,  $\gamma_{xy}$  and  $\tau_{xy}$ . Let  $\theta$  be a parameter of dimension  $d_\theta \leq N^{\text{mkts}} \times |\mathcal{X}| \times |\mathcal{Y}|$ , and assume that the functional form  $\alpha_{xy}^\theta$ ,  $\gamma_{xy}^\theta$  and  $\tau_{xy}^\theta$ . Then the identified set for  $\theta$  is given by

$$\Theta = \left\{ \theta \in \mathbb{R}^{d_\theta} : \psi_{xy} \left( \frac{\ln \mu_{xy}^k - \ln \mu_{x0}^k - \alpha_{xy}^\theta}{\tau_{xy}^\theta}, \frac{\ln \mu_{xy}^k - \ln \mu_{0y}^k - \gamma_{xy}^\theta}{\tau_{xy}^\theta} \right) = 0 \right. \\ \left. \forall k \leq d_\theta, x \in \mathcal{X}, y \in \mathcal{Y} \right\}. \quad (4.10)$$

Estimation of  $\theta$  in the point-identified case is discussed in the next section.

## 5. COMPARATIVE STATICS AND WELFARE ANALYSIS

In this section we examine how changes in the exogenous parameters affect the matching numbers  $\mu_{xy}$  and the equilibrium utilities  $U_{xy}$  and  $V_{xy}$ .

The type of comparative statics we obtain lead to several applications of interest:

As a first application, we examine what happens under zero-sum adjustment, i.e. the policy intervention consisting in decreasing  $\alpha_{xy}$  by some amount  $\delta s_{xy} > 0$  and increasing  $\gamma_{xy}$  by the same amount.<sup>9</sup> In the TU setting, the Becker–Coase theorem argues that the equilibrium outcome  $(\mu, U, V)$  is left unchanged under this policy experiment—market wages will decrease by *exactly* the market value of the new mandatory perk, so that the policy intervention is Pareto neutral. In the more general ITU setting, we find that the welfare of the intended beneficiary side will not necessarily increase, although this is typically the objective of the policy intervention.

As second application, we look at how the equilibrium is affected by demographic shocks, namely a change in the distributions of characteristics of the populations. Finally, we examine the impact of a change in the transfer function.

**5.1. General Comparative Statics.** In this section, we provide the most general possible comparative static result, which allows us to predict the vector of change in  $\delta\mu_{xy}$  in the number of matched pairs at equilibrium as a function of

- the changes in the affinities,  $\delta\alpha_{xy}$  and  $\delta\gamma_{xy}$ ,
- the change in the number of men and women of each types,  $\delta n_x$  and  $\delta m_y$ , and
- the change in the transfer function  $\delta\Psi_{xy}$ .

From the expression of  $\delta\mu$ , we can recover the expression of the systematic utilities at equilibrium  $\delta U$  and  $\delta V$ . In order to do this, we shall need to *vectorize* elements such as  $\alpha_{xy}$ , meaning that we should consider  $\alpha_{xy}$  as a *doubly-indexed vector*, namely an element of  $\mathbb{R}^{\mathcal{X}\times\mathcal{Y}}$ , rather than as a matrix. That way, the partial derivatives  $\partial\mu/\delta\alpha$  should be thought

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<sup>9</sup>In a labor economics context, an example of such policy intervention is a law which compels employers to grant a mandatory perk, such as luncheon vouchers.

of as *doubly-indexed matrix*, whose  $(xy)$   $(x'y')$ -th entry, which is the element at line  $xy$  and column  $x'y'$ , is  $\partial\mu_{xy}/\delta\alpha_{x'y'}$ .

We introduce the following notations:

- $\partial_u\Psi$  (resp.  $\partial_v\Psi$ ) is the doubly-indexed matrix whose  $(xy)$   $(x'y')$ -th entry is  $\partial_u\Psi_{xy}$  if  $x = x'$  and  $y = y'$ , 0 otherwise:

$$\mathbb{1}(x = x') \mathbb{1}(y = y') \partial_u\Psi_{xy} \text{ (resp. } \mathbb{1}(x = x') \mathbb{1}(y = y') \partial_v\Psi_{xy}\text{)}.$$

- $D^2G^*$  (resp.  $D^2H^*$ ) is the doubly-indexed matrix whose  $(xy)$   $(x'y')$ -th entry is  $\partial^2G^*(\mu)/\partial\mu_{xy}\partial\mu_{x'y'}$  (resp.  $\partial^2H^*(\mu)/\partial\mu_{xy}\partial\mu_{x'y'}$ ).
- $\delta\mu$ ,  $\delta U$ ,  $\delta V$ ,  $\delta\alpha$  and  $\delta\gamma$  are the doubly-indexed vectors whose  $(xy)$ -th entry are respectively  $\delta\mu_{xy}$ ,  $\delta U_{xy}$ ,  $\delta V_{xy}$ ,  $\delta\alpha_{xy}$  and  $\delta\gamma_{xy}$
- $\delta\Psi$  is the doubly-indexed vector whose  $(xy)$ -th entry is  $\delta\Psi_{xy}$
- $\frac{\mu\delta n}{n}$  (resp.  $\frac{\mu\delta m}{m}$ ) is the doubly-indexed vector whose  $(xy)$ -th term is  $\mu_{xy}\delta n_x/n_x$  (resp.  $\mu_{xy}\delta m_y/m_y$ )

Using these notations, we now state and prove our general comparative static. In the next subsections, we clarify the economic content of this result by specializing to different dimensions.

**Theorem 4.** *Under Assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique (this is in particular the case under additional assumption 2'). In this case, let  $(\mu, U, V)$  be the unique equilibrium outcome. Assume that  $\alpha$ ,  $\gamma$ ,  $n$ ,  $m$ , and  $\Psi$  are respectively changed by some infinitesimal quantities  $\delta\alpha$ ,  $\delta\gamma$ ,  $\delta n$ ,  $\delta m$ , and  $\delta\Psi$ , respectively. Then:*

(i) *The change in  $\mu$  is given by*

$$\begin{aligned} \delta\mu &= (\partial_u\Psi D^2G^* + \partial_v\Psi D^2H^*)^{-1} \\ &\cdot \left[ \partial_u\Psi \left( \delta\alpha + D^2G^* \frac{\mu\delta n}{n} \right) + \partial_v\Psi \left( \delta\gamma + D^2H^* \frac{\mu\delta m}{m} \right) - \delta\Psi \right] \end{aligned} \quad (5.1)$$

(ii) *The changes in  $U$  and  $V$  are given as a function of  $\delta\mu$  and  $\delta n$  by*

$$\delta U = (D^2G^*) \left( \delta\mu - \frac{\mu\delta n}{n} \right) \text{ and } \delta V = (D^2H^*) \left( \delta\mu - \frac{\mu\delta m}{m} \right). \quad (5.2)$$

In the ITU-Logit case, Theorem 4 takes the following form.

**Corollary 1.** Under Assumptions 1, 2', and 3, (5.1) becomes

$$\delta\mu = A^{-1}\delta z \quad (5.3)$$

where:

- $\delta z_{xy}$  is the doubly-indexed vector whose  $(xy)$ -th entry is

$$\partial_u \Psi_{xy} \left( \frac{\delta n_x}{\mu_{x0}} - \delta\alpha_{xy} \right) + \partial_v \Psi_{xy} \left( \frac{\delta m_y}{\mu_{0y}} - \delta\gamma_{xy} \right) - \delta\Psi_{xy}$$

- $A$  is the doubly-indexed matrix whose  $(xy)$   $(x'y')$ -th entry is

$$\frac{\partial_u \Psi_{xy} + \partial_v \Psi_{xy}}{\mu_{xy}} \mathbb{1}(x = x', y = y') + \frac{\partial_u \Psi_{xy}}{\mu_{x0}} \mathbb{1}(x = x') + \frac{\partial_v \Psi_{xy}}{\mu_{0y}} \mathbb{1}(y = y').$$

Meanwhile, (5.2) becomes

$$\delta U_{xy} = \frac{\delta\mu_{xy}}{\mu_{xy}} + \sum_{y'} \frac{\delta\mu_{xy'}}{\mu_{x0}} - \frac{\delta n_x}{\mu_{x0}}, \text{ and} \quad (5.4)$$

$$\delta V_{xy} = \frac{\delta\mu_{xy}}{\mu_{xy}} + \sum_{x'} \frac{\delta\mu_{x'y}}{\mu_{0y}} - \frac{\delta m_y}{\mu_{0y}}. \quad (5.5)$$

To clarify the structure of our comparative statics, we look at the case where there is no observable heterogeneity.

**Example 1.** Assume that there is only one type of man and one type of woman. In this case, our notations can be simplified: the number of married individuals is denoted  $\mu$ ; the number of single men is  $n - \mu$ ; and single women is  $m - \mu$ .

If we maintain assumptions 1, 2', and 3 and assume the solution of (3.6) exists and is unique, then the systematic part of the equilibrium utilities of married men and women are  $U = \log \mu / (n - \mu)$  and  $V = \log \mu / (m - \mu)$ , and  $D^2 G^* = \frac{n}{\mu(n-\mu)}$  and  $D^2 H^* = \frac{m}{\mu(m-\mu)}$ . Then, the equation of the model becomes  $\Psi(\log(\mu/(n - \mu)) - \alpha, \log(\mu/(m - \mu)) - \gamma) = 0$ , and Equation (5.1) simplifies to

$$\delta\mu = \frac{\partial_u \Psi \left( \delta\alpha + \frac{\delta n}{n-\mu} \right) + \partial_v \Psi \left( \delta\gamma + \frac{\delta m}{m-\mu} \right) - \delta\Psi}{\partial_u \Psi \frac{n}{\mu(n-\mu)} + \partial_v \Psi \frac{m}{\mu(m-\mu)}} \quad (5.6)$$

while the changes in  $U$  and  $V$  are given as a function of  $\delta\mu$  and  $\delta n$  by

$$\delta U = \frac{n\delta\mu - \mu\delta n}{\mu(n - \mu)} \text{ and } \delta V = \frac{m\delta\mu - \mu\delta m}{\mu(m - \mu)}. \quad (5.7)$$

In Appendix A, we give a slightly more complex example with one type of man and two types of women.

**5.2. Change in affinities.** We now focus on the effect of changes in the affinities  $\alpha$  and  $\gamma$  on the equilibrium, keeping  $n$ ,  $m$  and  $\Psi$  constant. The underlying policy question is how a change in the affinities will affect equilibrium outcomes: To what extent a policy intervention in a matching market can be undone by the invisible hand? In the case with Transferable Utility, this question is partially answered by the celebrated Becker–Coase theorem, which the following corollary to Theorem 4 recovers and extends.

**Corollary 2** (Principle of Unintended Consequences). *Maintain assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique. When the population numbers  $n$ ,  $m$  and the transfer function  $\Psi$  do not vary:*

(i) *Formula (5.1) becomes*

$$\delta\mu = (\partial_u\Psi D^2G^* + \partial_v\Psi D^2H^*)^{-1} (\partial_u\Psi\delta\alpha + \partial_v\Psi\delta\gamma).$$

(ii) *Under transferable utility, i.e when  $\Psi$  is given by (2.3), formula (5.1) becomes*

$$\delta\mu = (D^2G^* + D^2H^*)^{-1} (\delta\alpha + \delta\gamma).$$

**Corollary 3** (Becker–Coase Theorem). *Under the assumptions of Corollary 2, and under transferable utility and a zero-sum pre-equilibrium transfer, we have  $\delta\alpha_{xy} = -\delta s_{xy}$  and  $\delta\gamma_{xy} = \delta s_{xy}$ , so that  $\delta\mu_{xy} = 0$ ,  $\delta U_{xy} = 0$ , and  $\delta V_{xy} = 0$ .*

The Becker–Coase theorem can therefore be thought of as a “weak”, or “neutral” version of the principle of *unintended consequences* popularized by Merton (1936): The social policy designer intends to improve the welfare of one side of the market, say women, by imposing an exogenous zero-sum transfer from types  $x$  to types  $y$ . But the consequence of the policy is not felt by market participants, as efficient rebargaining occurs immediately after the policy change, and the rebargaining process exactly offsets the welfare changes caused by the transfer policy. This neutrality result is, however, very particular to TU matching. In the ITU case, the effect is much more subtle, as we now show: zero-sum transfer policies can

have averse unintended consequences, a “failure of good intentions.” The following example explores the simplified case when there is no observable heterogeneity.

**Example 1 Continued.** *In the setting of Example 1, the comparative statics formula becomes*

$$\frac{\delta\mu}{\mu} = \left( \frac{n\partial_u\psi}{n-\mu} + \frac{m\partial_v\psi}{m-\mu} \right)^{-1} (\partial_u\psi\delta\alpha + \partial_v\psi\delta\gamma).$$

*In particular, when  $\delta\alpha = -\delta s$  and  $\delta\gamma = \delta s$ , we have*

$$\frac{d \log \mu}{ds} = \frac{-\partial_u\psi + \partial_v\psi}{\frac{n\partial_u\psi}{n-\mu} + \frac{m\partial_v\psi}{m-\mu}}, \quad \frac{dU}{ds} = \frac{n}{\mu(n-\mu)} \frac{d\mu}{ds} \quad \text{and} \quad \frac{dV}{ds} = \frac{m}{\mu(m-\mu)} \frac{d\mu}{ds}.$$

*Note that in the TU case,  $\partial_u\psi = \partial_v\psi = 1$ , and hence in this case,  $d\mu/ds = dU/ds = dV/ds = 0$ , so we recover the Becker–Coase theorem.*

*In the general ITU case, we note that  $\frac{dU}{ds}$  and  $\frac{dV}{ds}$  have the same sign, which is negative if  $\partial_u\psi > \partial_v\psi$ . As a result, a policy intervention intended to benefit one side of the market may end up hurting both sides of the market—an occurrence of **adverse consequences**.*

Another interesting case is when  $\partial_u\Psi$  is much larger than  $\partial_v\Psi$ , i.e.  $\partial_u\Psi \gg \partial_v\Psi$ . This corresponds to the case where one side is very inelastic.

**Example 2.** Maintain assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique, and assume  $\partial_u\Psi \gg \partial_v\Psi$ . Then the formulae in Theorem 4 becomes

$$\delta\mu = (D^2G)(\delta\alpha), \quad \delta U = \delta\alpha, \quad \text{and} \quad \delta V = (D^2H^*)(D^2G)(\delta\alpha)$$

in particular, under a zero-sum pre-equilibrium transfer,  $\delta\alpha_{xy} = -\delta s_{xy}$  and  $\delta\gamma_{xy} = \delta s_{xy}$ , so that  $\delta U = -\delta s$  and  $\delta V = -(D^2H^*)(D^2G)\delta s$ .

Akin to a tax incidence result, this result shows that the policy has unambiguous consequences on one side of the market (here, men), but the consequences for the other side are rather unexpected, and can go in the opposite direction as intended. The result has an interesting interpretation. For instance, when the men’s side is completely inelastic ( $\partial_u\Psi \gg \partial_v\Psi$ ), any change in  $\alpha_{xy}$  will be fully appropriated by  $x$ ; the equilibrium matching  $\mu$  will adjust accordingly, and the utility change in  $V$  will adjust to the change in  $\mu$ .

**5.3. Change in Populations.** We now focus on the effect of a change in the population on the equilibrium outcome, assuming that the affinity parameters  $\alpha$  and  $\gamma$ , as well as the transfer function  $\Psi$ , remain constant. As Becker (1991, pp. 120–122) puts it:

“[...] An increase in the number of men of a particular quality tends to lower the incomes of all men and raise those of all women because of the competition in the marriage market between men and women of different qualities.”

We formalize this Beckerian intuition in the following result.

**Corollary 4** (Effects of Competition). *Maintain assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique. Assume the affinity parameters  $\alpha$  and  $\gamma$ , and the transfer function  $\Psi$  do not vary. When the population numbers vary by  $\delta n$  and  $\delta m$ , the variations in the numbers of matches  $\delta\mu$  are given by*

$$\delta\mu = (\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*)^{-1} \left[ \partial_u \Psi \left( D^2 G^* \frac{\mu \delta n}{n} \right) + \partial_v \Psi \left( D^2 H^* \frac{\mu \delta m}{m} \right) \right].$$

In particular, under transferable utility, i.e. when  $\Psi$  is given by (2.3), formula (5.1) becomes

$$\delta\mu = (D^2 G^* + D^2 H^*)^{-1} \left[ D^2 G^* \frac{\mu \delta n}{n} + D^2 H^* \frac{\mu \delta m}{m} \right].$$

**Example 1 Continued.** *In the setting of Example 1, the comparative statics formula becomes*

$$\delta\mu = \mu \frac{\partial_u \Psi (m - \mu) \delta n + \partial_v \Psi (n - \mu) \delta m}{\partial_u \Psi (m - \mu) n + \partial_v \Psi (n - \mu) m}.$$

We have

$$\frac{\delta\mu}{\mu} = \theta \frac{\delta n}{n} + (1 - \theta) \frac{\delta m}{m},$$

where we take

$$\theta \equiv \frac{\partial_u \Psi (m - \mu) n}{\partial_u \Psi (m - \mu) n + \partial_v \Psi (n - \mu) m}. \quad (5.8)$$

Thus, we can compute that

$$\delta U = \frac{n(1 - \theta)}{n - \mu} \left( \frac{\delta m}{m} - \frac{\delta n}{n} \right), \text{ and } \delta V = \frac{m\theta}{m - \mu} \left( \frac{\delta n}{n} - \frac{\delta m}{m} \right),$$

from which we see that if  $\frac{\delta m}{m} > \frac{\delta n}{n}$  (i.e., if the women's relative increase in population is larger than the men's), then the systematic utility of men increases, and the systematic utility of women decreases. (The converse is also true.)

Assume the affinity parameters  $\alpha$  and  $\gamma$ , and the transfer function  $\Psi$  do not vary. Let

$$u_x = G_x(U) \quad \text{and} \quad v_y = H_y(V)$$

be the average welfare of a man of type  $x$  and a woman of type  $y$ .

**Corollary 5** (Unexpected Symmetry). *Maintain Assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique. Assume that the population numbers vary by  $\delta n$  and  $\delta m$ . Then the “unexpected symmetry” result of Decker et al. (2012) holds within men and within women, namely*

$$\frac{\partial u_x}{\partial n_{x'}} = \frac{\partial u_{x'}}{\partial n_x} \quad \text{and} \quad \frac{\partial v_y}{\partial m_{y'}} = \frac{\partial v_{y'}}{\partial m_y};$$

however, symmetry does not necessarily hold across men and women, i.e.

$$\frac{\partial u_x}{\partial m_y} \quad \text{does not always coincide with} \quad \frac{\partial v_y}{\partial n_x},$$

even though it does in the TU case.

In the logit case (i.e., under Assumption 2'), we have  $u_x = -\log \frac{\mu_{x0}}{n_x}$ , so that

$$\frac{1}{\mu_{x0}} \frac{\partial \mu_{x0}}{\partial n_{x'}} = \frac{1}{\mu_{x'0}} \frac{\partial \mu_{x'0}}{\partial n_x} \quad \text{and} \quad \frac{1}{\mu_{0y}} \frac{\partial \mu_{0y}}{\partial m_{y'}} = \frac{1}{\mu_{0y'}} \frac{\partial \mu_{0y'}}{\partial m_y},$$

while

$$\frac{1}{\mu_{x0}} \frac{\partial \mu_{x0}}{\partial m_y} \neq \frac{1}{\mu_{0y}} \frac{\partial \mu_{0y}}{\partial n_x}.$$

**5.4. Change in Transfer Function.** Finally, we investigate the effect on equilibrium when the transfer function changes.

**Corollary 6.** *Maintain Assumptions 1, 2, and 3, and assume the solution of (3.6) exists and is unique. When the affinity parameters  $\alpha$  and  $\gamma$ , and the population numbers  $n$  and  $m$  are fixed, but when the transfer function varies by  $\delta \Psi$ , the variation in the numbers of matches  $\delta \mu$  is given by*

$$\delta \mu = (\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*)^{-1} \delta \Psi.$$

**Example 1 Continued.** *Once again, in the setting of Example 1, we have  $D^2G^* = \frac{n}{\mu(n-\mu)}$  and  $D^2H^* = \frac{m}{\mu(m-\mu)}$ . Then, the formula of Corollary 6 becomes*

$$\delta\mu = \frac{\mu(n-\mu)(m-\mu)\delta\Psi}{\partial_u\Psi n(m-\mu) + \partial_v\Psi(n-\mu)m}.$$

*This yields*

$$\delta U = \theta \frac{\delta\Psi}{\partial_u\Psi} \text{ and } \delta V = (1-\theta) \frac{\delta\Psi}{\partial_v\Psi},$$

*where  $\theta$  as defined in (5.8).*

## 6. DISCUSSION AND PERSPECTIVES

We have introduced an empirical framework for ITU matching with unobserved heterogeneity in tastes. Our framework includes as special cases the classic fully- and non-transferable utility models, collective models, and settings with taxes on transfers, dead-weight losses, and risk aversion. We characterized the equilibrium and identification conditions, and derived comparative statics.

Our equilibrium existence result allows general transfer functions  $\Psi$  (modulo the regularity conditions in Assumption 1), but requires a specific assumption on the form of heterogeneity (Assumption 2'). On a mathematical level, we would like to understand the weakest conditions on the transfer function  $\Psi$  and the distributions of heterogeneity  $\mathbf{P}_x$  and  $\mathbf{Q}_y$  under which the fundamental matching equation

$$\Psi(\nabla G^* - \alpha, \nabla H^* - \gamma) = 0$$

has a solution  $\mu \in \mathcal{M}$ . Galichon and Salanié (2014) allow for a general heterogeneity structure in the TU case, while Galichon and Hsieh (2014) do so in the NTU case.

A natural extension of our model is the model where some agents are forced to match, and the number of men and women coincide. In this case, the marginal constraint for  $\mu$  yields a *nonlinear Schrödinger system*. The set of solutions  $(a, b)$  is a manifold  $S$  of dimension 1, which has the property that there is a total order on  $\{(a, -b) : (a, b) \in S\}$  (see Carlier and Galichon (2014)).

Another natural extension incorporates many-to-one matching problems. In this context, a firm of type  $x \in \mathcal{X}$  matches with an (ordered) set of workers  $C = (y_1, \dots, y_p)$ , so that the

firm gets  $U_{xC}$  and the worker of type  $y_k$  gets  $V_{xC}^k$ , where these quantities are related by an extension of our transfer function

$$\Psi_{xC}(U_{xC}, V_{xC}^1, \dots, V_{xC}^p) = 0. \quad (6.1)$$

In current work, we are investigating an extension to Theorems 1 and 3 to this setting.

More far-reaching extensions to general network flow problems can be formulated. This is interesting in part because of the link between matching and hedonic models (Ekeland et al. (2006), Heckman et al. (2010)), which allows to apply some of the ideas in the present paper to consumer demand problems. We intend to explore these extensions in the future.

Beyond the class of problems investigated in the current paper, the methods developed here, based on fixed point theorems for isotone functions, may be more broadly applicable. In particular, they may be a relevant tool for the investigation of matching problems with peer effects put forward by Mourifié and Siow (2014).

On an empirical level, the present contribution will hopefully help bridge the gap between the empirical literature on matching with and without transferable utility. While the approach presented here is purely structural in nature, one could possibly combine the strategy here with a reduced-form approach based on shocks to matching patterns, like changes in divorce laws. The theory we present predicts that the effect of changes in the legal framework of divorce depends on the transferability parameter; hence, one could potentially use divorce law natural experiments, along with the comparative statics derived in Section 5, to make inference regarding the transferability parameter.

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#### APPENDIX A. COMPARATIVE STATICS WITH ONE MAN AND TWO WOMEN

Assume that there is one type of man  $\mathcal{X} = \{1\}$  and two types of women  $\mathcal{Y} = \{1, 2\}$ . Then  $G^*(\mu) = \mu_{11} \log \frac{\mu_{11}}{n_1} + \mu_{12} \log \frac{\mu_{12}}{n_1} + \mu_{10} \log \frac{\mu_{10}}{n_1}$ , thus  $U_{11} = \log \frac{\mu_{11}}{\mu_{10}}$  and  $U_{12} = \log \frac{\mu_{12}}{\mu_{10}}$ , so that

$$D^2 G^* = \begin{pmatrix} \frac{\mu_{11} + \mu_{10}}{\mu_{11} \mu_{10}} & \frac{1}{\mu_{10}} \\ \frac{1}{\mu_{10}} & \frac{\mu_{12} + \mu_{10}}{\mu_{12} \mu_{10}} \end{pmatrix}$$

, where  $\mu_{10} = n_1 - \mu_{11} - \mu_{12}$ .

Similarly,  $H^*(\mu) = \mu_{11} \log \frac{\mu_{11}}{m_1} + \mu_{01} \log \frac{\mu_{01}}{m_1} + \mu_{12} \log \frac{\mu_{12}}{m_2} + \mu_{02} \log \frac{\mu_{02}}{m_2}$ , thus  $V_{11} = \log \frac{\mu_{11}}{\mu_{01}}$  and  $V_{12} = \log \frac{\mu_{12}}{\mu_{02}}$ ; thus,

$$D^2 H^* = \begin{pmatrix} \frac{\mu_{11} + \mu_{01}}{\mu_{11} \mu_{01}} & 0 \\ 0 & \frac{\mu_{12} + \mu_{02}}{\mu_{12} \mu_{02}} \end{pmatrix},$$

where  $\mu_{01} = m_1 - \mu_{11}$  and  $\mu_{02} = m_2 - \mu_{12}$ .

In this case, we have

$$\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^* = \begin{pmatrix} \partial_u \Psi_{11} \frac{\mu_{11} + \mu_{10}}{\mu_{11} \mu_{10}} + \partial_v \Psi_{11} \frac{\mu_{11} + \mu_{01}}{\mu_{11} \mu_{01}} & \partial_u \Psi_{11} \frac{1}{\mu_{10}} \\ \partial_u \Psi_{12} \frac{1}{\mu_{10}} & \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12} \mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12} \mu_{02}} \end{pmatrix};$$

hence, we have

$$(\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*)^{-1} = d^{-1} \begin{pmatrix} \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12} \mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12} \mu_{02}} & -\partial_u \Psi_{11} \frac{1}{\mu_{10}} \\ -\partial_u \Psi_{12} \frac{1}{\mu_{10}} & \partial_u \Psi_{11} \frac{\mu_{11} + \mu_{10}}{\mu_{11} \mu_{10}} + \partial_v \Psi_{11} \frac{\mu_{11} + \mu_{01}}{\mu_{11} \mu_{01}} \end{pmatrix},$$

where  $d \equiv \det(\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*)$  is given by

$$d = \left( \partial_u \Psi_{11} \frac{\mu_{11} + \mu_{10}}{\mu_{11} \mu_{10}} + \partial_v \Psi_{11} \frac{\mu_{11} + \mu_{01}}{\mu_{11} \mu_{01}} \right) \left( \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12} \mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12} \mu_{02}} \right) - \partial_u \Psi_{12} \frac{1}{\mu_{10}} \partial_u \Psi_{11} \frac{1}{\mu_{10}}.$$

In particular, if the number of women of type 1 (i.e.,  $m_1$ ) changes, we have:

$$\begin{aligned}\delta\mu &= (\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*)^{-1} \begin{bmatrix} \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{11}\mu_{01}} \frac{\mu_{11} \delta m_1}{m_1} \right) \\ 0 \end{bmatrix} \\ &= d^{-1} \begin{bmatrix} \left( \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12}\mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12}\mu_{02}} \right) \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{01}} \frac{\delta m_1}{m_1} \right) \\ -\partial_u \Psi_{12} \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{10}\mu_{01}} \frac{\delta m_1}{m_1} \right) \end{bmatrix}.\end{aligned}$$

We then compute that

$$\begin{aligned}\delta\mu - \mu \frac{\delta m}{m} &= d^{-1} \begin{bmatrix} \left( \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12}\mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12}\mu_{02}} \right) \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{01}} \frac{\delta m_1}{m_1} \right) - d\mu_{11} \frac{\delta m_1}{m_1} \\ -\partial_u \Psi_{12} \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{10}\mu_{01}} \frac{\delta m_1}{m_1} \right) \end{bmatrix} \\ &= -d^{-1} \begin{bmatrix} \left( \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12}\mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12}\mu_{02}} \right) \partial_u \Psi_{11} \left( \frac{\mu_{11} + \mu_{10}}{\mu_{10}} \frac{\delta m_1}{m_1} \right) + \frac{\partial_u \Psi_{11} \partial_u \Psi_{12}}{\mu_{10}\mu_{10}} \mu_{11} \frac{\delta m_1}{m_1} \\ \partial_u \Psi_{12} \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{10}\mu_{01}} \frac{\delta m_1}{m_1} \right) \end{bmatrix}.\end{aligned}$$

From (5.2), we then find that

$$\begin{aligned}\delta V &= D^2 H^* \left( \delta\mu - \mu \frac{\delta m}{m} \right) \\ &= -d^{-1} \begin{bmatrix} \frac{\mu_{11} + \mu_{01}}{\mu_{11}\mu_{01}} \left( \left( \partial_u \Psi_{12} \frac{\mu_{12} + \mu_{10}}{\mu_{12}\mu_{10}} + \partial_v \Psi_{12} \frac{\mu_{12} + \mu_{02}}{\mu_{12}\mu_{02}} \right) \partial_u \Psi_{11} \left( \frac{\mu_{11} + \mu_{10}}{\mu_{10}} \frac{\delta m_1}{m_1} \right) + \frac{\partial_u \Psi_{11} \partial_u \Psi_{12}}{\mu_{10}\mu_{10}} \mu_{11} \frac{\delta m_1}{m_1} \right) \\ \frac{\mu_{12} + \mu_{02}}{\mu_{12}\mu_{02}} \left( \partial_u \Psi_{12} \partial_v \Psi_{11} \left( \frac{\mu_{11} + \mu_{01}}{\mu_{10}\mu_{01}} \frac{\delta m_1}{m_1} \right) \right) \end{bmatrix}.\end{aligned}\tag{A.1}$$

From (A.1), we see that the systematic utility of each women decreases following the increase in  $m_1$ :  $\delta V_{11} < 0$  and  $\delta V_{12} < 0$ .

## APPENDIX B. PROOFS OMITTED FROM THE MAIN TEXT

*Proof of Theorem 1.* Parts (i) and (ii) of this theorem follows by noticing that inverting (3.5) yields (3.7), which, combined with (3.4), yields (3.6). The proof of (iii) follows from the fact that if  $u_i$  and  $v_j$  are given by (3.8), then for all  $i$  and  $j$  such that  $x_i = x$  and  $y_j = y$ , inequalities  $u_i \geq U_{xy} + \varepsilon_{iy}$  and  $v_j \geq V_{xy} + \eta_{jx}$  hold, which implies

$$\Psi_{xy} \left( u_i - \alpha_{x_i y_j} - \varepsilon_{iy}, v_j - \gamma_{x_i y_j} - \eta_{jx} \right) \geq \Psi_{xy} (U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}) = 0$$

while this inequality holds as an equality if  $i$  and  $j$  are matched. ■

The proof of Theorem 2 is based on the following Lemma stating the properties of  $M_{xy}$ .

**Lemma 1.** *Under Assumption 1, and for every pair  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ :*

(i) *For each  $a, b > 0$ , equation  $\Psi_{xy}(\log M - \log a, \log M - \log b) = 0$  has a unique solution  $M > 0$ . This defines implicitly a map  $M_{xy}(a, b) \equiv M$  from  $(0, \infty)^2$  into  $(0, \infty)$ .*

(ii) *Map  $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$  is continuous.*

(iii) *Map  $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$  is weakly isotone, i.e. if  $a \leq a'$  and  $b \leq b'$ , then  $M_{xy}(a, b) \leq M_{xy}(a', b')$ .*

(iv) *For each  $a > 0$ ,  $\lim_{b \rightarrow 0^+} M_{xy}(a, b) = 0$ , and for each  $b > 0$ ,  $\lim_{a \rightarrow 0^+} M_{xy}(a, b) = 0$ .*

*Proof of Lemma 1.* (i) The three assumptions in 1 imply that for any  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$  and reals  $u$  and  $v$ , the map  $t \rightarrow \Psi_{xy}(u + t, v + t)$  is continuous, strictly increasing and changes sign when  $t$  ranges between  $-\infty$  and  $+\infty$ . By the Intermediate Value theorem, there exists a unique value  $L_{xy}(a, b)$  such that  $\Psi_{xy}(L_{xy}(a, b) - \log a, L_{xy}(a, b) - \log b) = 0$ . Setting  $M_{xy}(a, b) := \exp L_{xy}(a, b)$  yields the desired result.

(ii) Let  $(a, b) \in (0, +\infty)^2$ , and consider a sequence  $(a_n, b_n) \rightarrow (a, b)$ , which can be assumed bounded away from 0. Let  $L = \log M_{xy}(a, b)$  and  $L_n = \log M_{xy}(a_n, b_n)$ . One has  $\Psi_{xy}(L_n - \log a_n, L_n - \log b_n) = 0$ . By 1 (d),  $L_n$  cannot have a diverging subsequence; hence it is bounded. Take a converging subsequence of  $L_n$  and call  $L'$  its limit. By 1 (a) (continuity of  $\Psi$ ), one has  $\Psi_{xy}(L' - \log a, L' - \log b) = 0$ , and by uniqueness,  $L' = L$ . Therefore,  $L_n$  is bounded and all its subsequences converge to  $L$ , thus  $L_n \rightarrow L$ , which establishes continuity of  $M_{xy}$ .

(iii) Assume  $a \leq a'$  and  $b \leq b'$ . Then by Assumption 1, 1 (b), one has

$$\Psi_{xy}(\log M - \log a', \log M - \log b') \leq \Psi_{xy}(\log M - \log a, \log M - \log b),$$

thus  $\Psi_{xy}(\log M(a, b) - \log a', \log M(a, b) - \log b') \leq 0 =$ , hence  $M(a, b) \leq M(a', b')$ .

(iv) For  $a > 0$ , let  $b_n$  be a decreasing sequence converging to 0, and let  $L_n = \log M_{xy}(a, b_n)$ .  $L_n$  is nondecreasing; let  $L$  be its limit. Assume  $L > -\infty$ . Then  $u_n = L_n - \log a$  is bounded, while  $v_n = L_n - \log b_n \rightarrow +\infty$ . Then  $\Psi_{xy}(u_n, v_n) = 0$  comes in contradiction with Assumption 1, 1 (c). Thus  $L = -\infty$ , and thus  $M_{xy}(a, b_n) \rightarrow 0^+$ . ■

We may now turn to the proof of Theorem 2.

*Proof of Theorem 2.* The proof of existence is essentially an application of Tarski's fixed point theorem; we provide an explicit proof for concreteness. We need to prove that the construction of  $\mu_{x0}^{2t+1}$  and  $\mu_{0y}^{2t+2}$  at each step is well defined. Consider step  $2t + 1$ . For each  $x \in \mathcal{X}$ , the equation to solve is

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$

but the right handside is a continuous and increasing function of  $\mu_{x0}$ , tends to 0 when  $\mu_{x0} \rightarrow 0$  and tends to  $+\infty$  when  $\mu_{x0} \rightarrow +\infty$ . Hence  $\mu_{x0}^{2t+1}$  is well defined and belongs in  $(0, +\infty)$ , and let us denote

$$\mu_{x0}^{2t+1} = F_x(\mu_{0.}^{2t})$$

and clearly,  $F$  is anti-isotone, meaning that  $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$  for all  $y \in \mathcal{Y}$  implies  $F_x(\tilde{\mu}_{0.}^{2t}) \leq F_x(\mu_{0.}^{2t})$  for all  $x \in \mathcal{X}$ .

By the same token, at step  $2t + 2$ ,  $\mu_{0y}^{2t+2}$  is well defined in  $(0, +\infty)$ , and let us denote

$$\mu_{0y}^{2t+2} = G_y(\mu_{.0}^{2t+1})$$

where, similarly,  $G$  is anti-isotone. Thus

$$\mu_{0.}^{2t+2} = G \circ F(\mu_{0.}^{2t})$$

where  $G \circ F$  is isotone. But  $\mu_{0y}^2 \leq m_y = \mu_{0y}^0$  implies that  $\mu_{0.}^{2t+2} \leq G \circ F(\mu_{0.}^{2t})$ . Hence  $(\mu_{0.}^{2t+2})_{t \in \mathbb{N}}$  is a decreasing sequence, bounded from below by zero. As a result  $\mu_{0.}^{2t+2}$  converges. Letting  $\bar{\mu}_{0.}$  its limit, and letting  $\bar{\mu}_{.0} = F(\bar{\mu}_{0.})$ , it is not hard to see that  $(\bar{\mu}_{0x}, \bar{\mu}_{0y})$  is a solution to (4.7). ■

*Proof of Theorem 3.* Introduce map  $\zeta$  defined by

$$\zeta : (\mu_{x0}, \mu_{0y}) \rightarrow \begin{cases} \zeta_x = \sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} \\ \zeta_y = \sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{0y} \end{cases}$$

One has

$$D\zeta(\mu_{x0}, \mu_{0y}) = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

where:

- $A = (\partial \zeta_x / \partial \mu_{x'0})_{xx'} = \sum_{y' \in \mathcal{Y}} \partial_{\mu_{x0}} M_{xy'}(\mu_{x0}, \mu_{0y'}) + 1$  if  $x = x'$ , 0 otherwise.

- $B = (\partial\zeta_x/\partial\mu_{0y})_{xy} = \partial_{\mu_{0y}} M_{xy}(\mu_{x0}, \mu_{0y})$
- $C = (\partial\zeta_y/\partial\mu_{x0})_{yx} = \partial_{\mu_{x0}} M_{xy}(\mu_{x0}, \mu_{0y})$
- $D = (\partial\zeta_y/\partial\mu_{0y'})_{yy'} = \sum_{x' \in \mathcal{X}} \partial_{\mu_{0y}} M_{x'y}(\mu_{x'0}, \mu_{0y}) + 1$  if  $y = y'$ , 0 otherwise.

It is straightforward to show that the matrix  $D\zeta$  is dominant diagonal. A result from McKenzie (1960) states that a dominant diagonal matrix with positive diagonal entries is a P-matrix. Hence  $D\zeta(\mu_{x0}, \mu_{0y})$  is a P-matrix. Applying Theorem 4 in Gale and Nikaido (1965) it follows that  $\zeta$  is injective. ■

*Proof of Proposition 1.* The result is immediate as the restrictions of the model are exhausted by the set of equations (4.9). ■

*Proof of Theorem 4.* We shall first prove the second statement, and express  $\delta U$  as a function of  $\delta\mu$  and  $\delta n$ . One has  $U_{xy} = \partial_{\mu_{xy}} G^*$ , so

$$\delta U_{xy} = \sum_{x'} \partial_{n_{x'}} \partial_{\mu_{xy}} G^*(\mu) \delta n_{x'} + \sum_{x'y'} \partial_{\mu_{xy}} \partial_{\mu_{x'y'}} G^*(\mu) \delta \mu_{x'y'}.$$

From Expressions (3.1) and (3.2), it follows that  $G^*(\mu) = \sum_x n_x G_x^*(\mu/n_x)$ , thus

$$\begin{aligned} \partial_{n_x} G^*(\mu) &= G_x^*(\mu/n_x) - \sum_y \frac{\mu_{xy}}{n_x} \partial_{\mu_{y|x}} G_x^*(\mu/n_x) = G_x^*(\mu/n_x) - \sum_y \frac{\mu_{xy}}{n_x} U_{xy} \\ &= -G_x(U) \end{aligned}$$

hence

$$\begin{aligned} \partial_{n_{x'}} \partial_{\mu_{xy}} G^*(\mu) &= -\partial_{\mu_{xy}} G_{x'}(U) = -\sum_z (\partial G_{x'}/\partial U_{x'z}) (\partial U_{x'z}/\partial \mu_{xy}) \\ &= -\sum_z \mu_{z|x} \frac{\partial^2 G^*}{\partial \mu_{xz} \partial \mu_{xy}} \text{ if } x' = x \\ &= 0 \text{ otherwise} \end{aligned}$$

Therefore,

$$\delta U_{xy} = \sum_z \frac{\partial^2 G^*(\mu)}{\partial \mu_{xy} \partial \mu_{xz}} \left( \frac{n_x \delta \mu_{xz} - \mu_{xz} \delta n_x}{n_x} \right),$$

and the formula for  $\delta V_{xy}$  is obtained by similar means. Hence (5.2) follow.

We can now use (5.2) to obtain a linearization of (3.6). We get

$$\partial_u \Psi_{xy} (\delta U_{xy} - \delta \alpha_{xy}) + \partial_v \Psi_{xy} (\delta V_{xy} - \delta \gamma_{xy}) + \delta \Psi_{xy=0}$$

thus, expressing  $\delta U$  and  $\delta V$  as a function of  $\delta \mu$  obtains

$$\begin{aligned} 0 &= \partial_u \Psi_{xy} \left( \sum_z \frac{\partial^2 G^*}{\partial \mu_{xz} \partial \mu_{xy}} \left( \frac{n_x \delta \mu_{xz} - \mu_{xz} \delta n_x}{n_x} \right) - \delta \alpha_{xy} \right) \\ &\quad + \partial_v \Psi_{xy} \left( \sum_t \frac{\partial^2 H^*}{\partial \mu_{ty} \partial \mu_{xy}} \left( \frac{m_y \delta \mu_{ty} - \mu_{ty} \delta m_y}{m_y} \right) - \delta \gamma_{xy} \right) \\ &\quad + \delta \Psi_{xy} \end{aligned}$$

hence

$$\begin{aligned} &\partial_u \Psi_{xy} \left( \sum_z \frac{\partial^2 G^*}{\partial \mu_{xz} \partial \mu_{xy}} \delta \mu_{xz} \right) + \partial_v \Psi_{xy} \left( \sum_t \frac{\partial^2 H^*}{\partial \mu_{ty} \partial \mu_{xy}} \delta \mu_{ty} \right) \\ &= \\ &\partial_u \Psi_{xy} \left( \delta \alpha_{xy} + \sum_z \frac{\partial^2 G^*}{\partial \mu_{xz} \partial \mu_{xy}} \frac{\mu_{xz} \delta n_x}{n_x} \right) + \partial_v \Psi_{xy} \left( \delta \gamma_{xy} + \sum_t \frac{\partial^2 H^*}{\partial \mu_{ty} \partial \mu_{xy}} \frac{\mu_{ty} \delta m_y}{m_y} \right) - \delta \Psi_{xy} \end{aligned}$$

which rewrites in a matrix way as

$$\begin{aligned} (\partial_u \Psi D^2 G^* + \partial_v \Psi D^2 H^*) \delta \mu &= (\partial_u \Psi) \left( \delta \alpha + (D^2 G^*) \left( \frac{\mu \delta n}{n} \right) \right) + \partial_v \Psi \left( \delta \gamma + (D^2 H^*) \left( \frac{\mu \delta m}{m} \right) \right) \\ &\quad - \delta \Psi \end{aligned}$$

that is (5.1). ■

*Proof of Corollary 1.* In the ITU-Logit case, one has  $U_{xy} = \log \mu_{xy} - \log \left( n_x - \sum_{y'} \mu_{xy'} \right)$ ,

thus

$$\delta U_{xy} = \frac{\delta \mu_{xy}}{\mu_{xy}} + \sum_{y'} \frac{\delta \mu_{xy'}}{\mu_{x0}} - \frac{\delta n_x}{\mu_{x0}},$$

and, similarly,

$$\delta V_{xy} = \frac{\delta \mu_{xy}}{\mu_{xy}} + \sum_{x'} \frac{\delta \mu_{x'y}}{\mu_{0y}} - \frac{\delta m_y}{\mu_{0y}}$$

and the linearization of Equation (3.6) yields

$$\begin{aligned}
0 &= \partial_u \Psi_{xy} \left( \frac{\delta \mu_{xy}}{\mu_{xy}} + \sum_{y'} \frac{\delta \mu_{xy'}}{\mu_{x0}} - \frac{\delta n_x}{\mu_{x0}} - \delta \alpha_{xy} \right) \\
&\quad + \partial_v \Psi_{xy} \left( \frac{\delta \mu_{xy}}{\mu_{xy}} + \sum_{x'} \frac{\delta \mu_{x'y}}{\mu_{0y}} - \frac{\delta m_y}{\mu_{0y}} - \delta \gamma_{xy} \right) \\
&\quad + \delta \Psi_{xy}
\end{aligned}$$

thus  $A\delta\mu = \delta z$  using the notation of the statement of the result. Expression (5.3) follows. ■

*Proof of Corollary 5.* We have

$$G_x(U) = -\partial_{n_x}(G^*(\mu)),$$

so that

$$\begin{aligned}
\partial_{n_{x'}} u_x &= \partial_{n_{x'}}(G_x(U)) = -\partial_{n_{x'}} \partial_{n_x}(G^*(\mu)) \\
&= -\partial_{n_x} \partial_{n_{x'}}(G^*(\mu)) = \partial_{n_x} G_{x'}(U) = \partial_{n_x} u_{x'}.
\end{aligned}$$

A similar formula holds on the other side of the market. However,  $\partial_{m_y} u_x = \partial_{m_y} G_x(U) = -\partial_{m_y} \partial_{n_x} G^*(\mu) = -\partial_{n_x} \partial_{m_y} G^*(\mu)$  has no reason to coincide with  $-\partial_{n_x} \partial_{m_y} H^*(\mu) = \partial_{n_x} H_y(V) = \partial_{n_x} v_y$ . The reason this is true in the TU case is that then  $G_x(U) = \partial_{n_x}(G(U) + H(V))$ , thus

$$\begin{aligned}
\partial_{m_y} u_x &= \partial_{m_y} G_x(U) = \partial_{m_y} \partial_{n_x}(G(U) + H(V)) \\
&= \partial_{n_x} \partial_{m_y}(G(U) + H(V)) = \partial_{n_x} H_y(V) = \partial_{n_x} v_y.
\end{aligned}$$

■

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