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Sovereign Default and Liquidity: The Case for a World Safe Asset[*]

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Abstract

We present a general equilibrium model of the world economy where sovereigns face idiosyncratic risks and can default on their debt. In this model, the world interest rate is determined through the global financial market equilibrium, the amount of safe asset is endogenous and determines international risk sharing. Non-trivial multiple equilibria naturally arise, due to the endogeneity of the interest rate. These equilibria can be ranked according to their aggregate welfare, such that equilibria with a higher quantity of safe assets correspond to higher welfare. Due to a shortage in the safe asset supply, even the equilibrium with the highest welfare is not constrained-efficient. Finally, we prove that a world fund issuing a safe asset can reach the constrained-efficient aggregate welfare. With a standard calibration, the size of the fund is found to be around 4.1% of the world GDP. Its relationship with the Special Drawing Rights of the IMF is discussed.

1 Introduction

The global economy experiences two related features. The first one is the pervasiveness of sovereign default. The largest default in history (by present value) was the 2012 Greek restructuring that covered more than € 200 bn of privately held debt (Tomz and Wright 2013). Debates about a larger restructuring of the Greek debt or about the default of another developed country indicate that sovereign default may reach another order of magnitude in the near future. Even

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if the default risk does not materialize, the discussion about its very possibility is a clear sign of the deterioration of the perceived safeness of public debt in some countries. The second feature affecting the global economy is the apparent shortage of safe assets. Observing the downward trend in the return on US public debt, Caballero and Farhi (2017), Barro and Mollerus (2016), and Hall (2016), among others, discuss the effects of a shortage of safe assets.[1] These two facts are obviously related, as sovereign default almost mechanically reduces the quantity of safe asset. In addition, the quantity of safe asset affects the level of the real interest rates and the default probability, through its effect on both the incentives to accumulate assets and on the opportunity cost to default.

The goal of this paper is to investigate the interactions between the quantity of safe assets and sovereign default in general equilibrium. We are interested in both positive and normative questions. What are the determinants of the quantity of safe assets when the sovereigns can choose to strategically default? Does the market economy generate too much safe asset, as a general equilibrium outcome? Should a world institution issue a safe asset? In which amount? This last question was, for instance, discussed in policy circles at the IMF before the decision to issue Special Drawing Rights.

To contribute to answer these questions, this paper presents a general equilibrium model of default, where sovereigns can borrow or lend on international markets, and possibly default on their debt, in order to smooth idiosyncratic income shocks. Elaborating on the Eaton and Gersovitz (1981) literature, the basic friction we consider is the lack of complete insurance market for country specific risk. Countries can only issue non-contingent claims, on which they can default. Equilibrium default can occur when the burden of the debt is high in bad times and when the opportunity cost of default is low. After the work of Aguiar and Gopinath (2006) and Arellano (2008) among others, this framework has become a benchmark to study sovereign default (see the recent exposition of Amador and Aguiar 2015). In this literature, the papers assume that the real interest rate is exogenous, so that they can investigate the amount of debt and the default decision in rich environments (see the literature review below). Our contribution is to endogenize the world interest rate by developing a tractable general equilibrium model.

We model the world economy as a continuum of countries borrowing and lending to each other to smooth idiosyncratic shocks. Countries can default on their debt depending on the

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[1]See also Gorton, Lewellen, and Metrick (2012) for a measure of the safe asset share in the US economy.

endogenous intertemporal cost of default. When countries default, they are temporarily excluded from financial markets. The world interest rate is determined by the international financial market clearing. An equilibrium in this economy is characterized by the default policy, the world wealth distribution and world interest rates. An alternative angle to consider this economy is to think about a Bewley economy where agents do not face exogenous credit constraint but can possibly default, if they find it optimal to do so. Surprisingly, the constrained-efficiency of market economies and of optimal policies in these environments have not been studied yet.[2] To do so, we use the strategy of Davila, Hong, Krusell, and Rios-Rull (2012), developed in incomplete-market economies without default, to derive the constrained-efficient allocation. We then analyze the properties of the market economy, and notably the welfare and quantity of safe asset, determining international risk sharing. We derive three main results.

The first result is that the interaction between incomplete markets and default generate multiple equilibria, differing in the quantity of the safe asset and interest rates. When the interest rate is high, the incentives to save for self-insurance are high. As countries save more, there is less default because the opportunity cost to default is high. Participating in financial markets is highly valued as risk sharing is important. A high supply of safe asset is consistent with a low default risk. On the contrary, with a smaller supply of safe assets available for self-insurance, there is more default in general equilibrium, as the relative gain to participating in financial markets is low. The multiple equilibria can be ranked according to their aggregate welfare. Equilibrium with less default unambiguously corresponds to higher welfare, compared to equilibrium with more defaults. The multiplicity of equilibria crucially thus depends on the endogeneity of the interest rate.

Second, we show that the market equilibrium with the highest welfare is still constrained-inefficient. The reason being that when markets are incomplete, prices do not convey the right incentives to issue safe assets. In other words, there is a pecuniary externality, implying a lack of risk sharing and a too small quantity of safe asset in the market economy. Sovereigns do not properly internalize in their net saving and default decisions the social benefits of their debt as a safe asset for other countries.

Third, the welfare in the constrained-efficient economy can be reached thanks to the intro-

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[2]Livshits, MacGee, and Tertilt (2007) and Chatterjee, Cordae, Nakajima and Rios-Rull (2007) have used these environments to compare the effect of different bankruptcy rules in quantitative setups. We instead simplify the environment to derive optimal policies in these environments.

duction of an international fund. This fund issues interest-bearing assets that are financed with the voluntarily contribution of member states. More precisely, the countries joining the fund receive a subsidy, while countries belonging to the fund pay a non state-contingent contribution to balance the budget of the fund. There is an equilibrium where all countries are either excluded from financial markets because they have defaulted or they all voluntarily contribute to the fund. In this case, there is full risk sharing among countries participating in financial markets, but default still occurs in general equilibrium. We provide a calibration of the model using both data on sovereign default and on "disaster event", to reproduce both the default probability and income fluctuations. The optimal quantity of assets issued by the fund represents around 4.1% of world GDP. The countries average contributions is equal to 0.22% of their own GDP, and the subsidy given by the the fund to joining countries is 3.49% of their GDP.

The fund and its asset supply are obviously reminiscent of the IMF Special Drawing Rights (SDRs). These SDRs were issued in the early 70s after a world discussion about the scarcity of safe stores of value (see Williamson 2009 for a short history). SDRs are interest-bearing assets, whose interest rate is weekly determined as the average interest rate on the money markets for a basket of currencies.[3] There are nevertheless two main differences between the assets issued by the fund in our model and SDRs. First, the interest rate on the asset issued by the fund in our model should be linked to the interest paid by government for long-maturities (which is the horizon for self-insurance) instead of the interest on money market as for SDRs. Second, the outstanding amounts differ substantially. For SDRs, the outstanding amount is smaller than 0.3% of the world GDP in 2016, while, with our calibration, the optimal issuance of world safe asset should correspond to 4.1% of world GDP.

This paper contributes to several strands of the literature. First, it belongs to the literature on the safe assets. This recent literature (Gorton, Lewellen, and Metrick 2012, Barro and Mollerus 2016, Hall 2016, and Caballero and Farhi 2017) studies the implication of the possible lack of safe assets. Another large literature investigates the determinants of the quantity of the world safe asset and of the interest rate (Mendoza and Quadrini 2009, Azzimonti, de Francesco, and Quadrini 2014). We introduce the interaction between sovereign default and the quantity of a safe asset in this literature. Recently, Farhi and Maggiori (2017) propose a model of the

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[3]See the IMF website for precise definition and the value of the interest rate on SDRs: `http://www.imf.org/external/np/fin/data/sdr_ir.aspx`.

international monetary system and study various market structures and frictions. We instead focus only on the optimality of the supply of the world safe asset. The gain of this strategy is to better understand market failures along this dimension and highlight new mechanisms, such as equilibrium multiplicity and the optimal size of a world fund.

Second, this paper belongs to the vast literature on sovereign default, which builds on Eaton and Gersovitz (1981) framework. This literature reproduces the empirical pattern of sovereign default in partial equilibrium (Aguiar and Gopinath 2006, Arellano 2008 among others, as well as the recent survey of Amador and Aguiar 2015). In this literature, our contribution is to investigate normative implications in general equilibrium and the possibility of equilibrium multiplicity.[4]

Our paper also belongs to the literature about default in general equilibrium. First, Zame (1993) and Dubey, Geanakoplos, and Shubik (2005) study in a two-period economy featuring incomplete markets the role of default, when default generates explicit costs. The literature on household default (Athreya 2002, Chatterjee, Cordae, Nakajima and Rios-Rull 2007, Livhists, MacGee and Tertilt 2007) has studied general equilibrium models. We focus on a simpler model to understand the distortions of the market economy, and to derive optimal policies.

Section 2 presents the environment. Section 3 solves for the market equilibrium and studies equilibrium multiplicity. Section 4 presents the constrained-efficient allocation. Section 5 shows that this allocation can be decentralized with a fund. Section 6 presents our numerical application and Section 7 concludes.

2 Environment

2.1 Set-up

Time is discrete $t = 1, \ldots, \infty$. The world economy is modeled as a continuum of small open economies. Countries are distributed in each period t according to a uniform distribution G over a segment I of length 1.[5] Each economy has a risky production technology and a benevolent government maximizes the utility on behalf of a unit mass of identical consumers. Countries

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[4]Mendoza and Yue 2012 endogenize the cost of default using a model of trade credit.

[5]This representation of the world economies follows Clarida (1990) or Bai and Zhang (2010). There is a literature about the applicability of the law of large number in continuum economies –see Feldman and Gilles (1985) and Green (1994) among others. We simply here assume that the law of large number applies.

face idiosyncratic production shocks, but there is no aggregate shock at the world level.

Each country has identical, additive, and time-separable preferences over streams of consumption $(c_t)_{t \geq 0}$ and labor supply $(l_t)_{t \geq 0}$. The period utility function over consumption c and labor supply l is assumed to be quasi-linear: $u(c) - l$, where the consumption utility function u is increasing, differentiable, and concave. Quasi-linearity simplifies the equilibrium structure and preserves the essence of default mechanisms. Hansen (1985) and Rogerson (1988) provide a microfoundation for the linear disutility in labor by assuming that labor is indivisible and that the planner allocates labor within the country through lotteries. The utility function is used by Lagos and Wright (2005) in a matching environment and by Scheinkman and Weiss (1986) and Challe, LeGrand, and Ragot (2013) in an incomplete-market environment without default, among others. In each country, the government maximizes the intertemporal welfare $\sum_{t=0}^{\infty} \beta^t (u(c_t) - l_t)$, where $\beta \in (0, 1)$ is the constant discount factor.

Countries face a productivity risk that can be neither avoided nor insured. The productivity status of a given country can be in one of two states, which are described as *productive* (state p) and *unproductive* (state u). When a country is productive, it has access to a linear production technology, which transforms l units of labor into l units of final goods. The country labor supply can in addition be freely adjusted in every period. When a country is unproductive, it is restricted to supply an amount $\bar{l} < 1$ of labor. This restriction will ensure that both the period utility level and the marginal utility of consumption for a country in a productive state are smaller than those of a country in an unproductive state.

The productivity status follows a first-order Markov chain with transition matrix $\Pi = \begin{bmatrix} \alpha & 1 - \alpha \\ 1 - \rho & \rho \end{bmatrix} \in (0, 1)^4$, where for example the probability $1 - \alpha$ is the probability to switch from state p in the current period to state u in the next one.

2.2 Financial markets and default

As is standard in the literature on sovereign default, following Eaton and Gersovitz (1981) and Bulow and Rogoff (1989), international financial markets are assumed to be plagued by two frictions. The first one is the incompleteness of insurance-markets. Countries cannot trade assets contingent on their next period productive status, but can only trade one-period bonds. Second, countries can default on their debt, but at the cost of a temporary exclusion from

financial markets. We assume that both the current productivity status and the asset volume for each country is observable when debt is sold or bought on international markets. If a country defaults on its debt, it is excluded from international markets and has to live in autarky. Even though excluded countries are prevented from saving or borrowing, they are affected by the productivity shock.

To make the model empirically relevant, we follow Arellano (2008) in introducing an additional cost of default. Countries in autarky have only access to a less efficient production technology, through which 1 unit of labor is only transformed into $\varphi \in (0, 1]$ units of consumption. Excluded productive countries have in every period the probability $\theta \in (0, 1)$ to reenter financial markets, while excluded unproductive agents cannot reenter financial markets. When reentering financial markers, the country is endowed with no financial wealth.

The debt is traded through a unit mass of competing risk-neutral financial intermediaries. Saving countries lend to financial intermediaries, whereas borrowing countries borrow from them. Financial intermediaries diversify their risk across countries and therefore act as devices pooling the idiosyncratic risk. As a consequence, in absence of aggregate risk, financial intermediaries can offer a risk-free rate to saving countries and will charge a credit risk premium to borrowers according to their default probability.

We denote as $q(B', s)$ the price of a claim on one unit of next period good for a country who chooses an amount of asset B' and which has the production status $s = \{p, u\}$. If $B' > 0$, the country is saving in a safe asset of price $q(B', s) = q$. The real interest rate on the safe asset is simply $1 + r \equiv \frac{1}{q}$. If $B' < 0$, the country is borrowing. The assumption of perfectly diversified intermediaries implies that the price $q(B', s)$ can be expressed as:

$$q(B', s) = q \times (1 - \delta(B', s)),$$

where $\delta(B', s)$ is the default probability of a country choosing a debt B' while its current status is $s \in \{p, u\}$. Both the price of a safe asset q and the default probability $\delta(B', s)$ will be general equilibrium endogenous outcomes. A country in state $s \in \{p, u\}$, endowed with the beginning-of-period wealth B, and trading an amount of debt B', and supplying a labor quantity l, will have a consumption equal to:

$$c = l + B - q(B', s) B'. \tag{1}$$

If the country is in a productive state, it can then freely adjust the labor effort l. If unproductive, the country will supply the fixed amount $l = \bar{l}$.

Since the default probability $\delta(B', s)$ is not necessarily continuous in the debt level B', agents will have to compare different debt levels corresponding to different prices. Furthermore, budget set may not be convex. There is no guarantee of equilibrium uniqueness in the general case, and we will show that it is not the case. However, although the problem is not convex, it can be written in a recursive form (see Stokey and Lucas 1989, Theorem 9.4), which will allow us to simplify the exposition and to derive necessary first-order conditions.

2.3 Welfare functions and financial market clearing

We define $V_s^o(B)$ as the value function of a country participating in financial market, starting the current period with the asset holding B, while being in the productivity status $s \in \{p, u\}$.[6] The superscript o denotes a participating country which has the *option* to default on B. The country decides then whether to default or repay its debts by choosing the option that is associated to the highest welfare. We denote $V_s^c(B)$ the value function of a country deciding to repay the debt B, while its status is s. The superscript c indicates that the country decide to *continue* to honor its debts. Similarly, the value function V_s^d is the value function of the country deciding to default and is independent of the defaulted amount. The superscript d denotes the *default* decision. The value function $V_s^o(B)$ is thus equal to the maximum between the value functions associated to debt repayment or to default. Formally:

$$\forall B, \forall s \in \{p, u\}, \quad V_s^o(B) = \max\{V_s^c(B), V_s^d\}. \tag{2}$$

Let us now turn to the expression of value functions V^d and V^c. Since defaulted countries face the probability θ to possibly reenter financial markets when productive with a zero wealth, the value function associated to default can be expressed as:

$$V_p^d = \max_l \ (u(\varphi l) - l) + \beta\alpha\left(\theta V_p^o(0) + (1-\theta)V_p^d\right) + \beta(1-\alpha)V_u^d, \tag{3}$$

$$V_u^d = u\left(\varphi\bar{l}\right) - \bar{l} + \beta(1-\rho)\left(\theta V_p^o(0) + (1-\theta)V_p^d\right) + \beta\rho V_u^d. \tag{4}$$

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[6]For the sake of simplicity, we follow the notations of Arellano (2008).

In these two equations; the country is excluded from financial markets, and is prevented from borrowing and saving. The whole production φl or $\varphi \bar{l}$ of the period is consumed within the period.

When the country participates in financial markets and chooses not to default in the current period, the value function V^c can be expressed as follows, for any debt endowment B:

$$V_p^c(B) = \max_{l,B'} \ u\left(l + B - q\left(B',s\right)B'\right) - l + \beta\left(\alpha V_p^o\left(B'\right) + (1-\alpha)V_u^o\left(B'\right)\right), \qquad (5)$$

$$V_u^c(B) = \max_{B'} \ u\left(\bar{l} + B - q\left(B',s\right)B'\right) - \bar{l} + \beta\left((1-\rho)V_p^o\left(B'\right) + \rho V_u^o\left(B'\right)\right), \qquad (6)$$

where we used the budget constraint (1) to substitute for the consumption expression. As the value functions V_s^c for $s = p, u$ are increasing in the wealth amount B, we can define the quantities $\overline{B}^s \in \mathbb{R} \cup \{-\infty\}$, $s = p, u$ as:

$$\overline{B}^s = \min_{B \in \mathbb{R}}\{B | V_s^c(B) \geq V_s^d\}, s = p, u. \qquad (7)$$

In words, the quantity $\overline{B}^s$ is the state-contingent threshold such that a country endowed with a wealth level above the threshold will decide not to default.

We can finally provide the expression for the clearing of financial markets. Since assets are in zero net supply and competition among financial intermediaries imply a zero-profit condition, the value of all net asset demands sums up to zero. Recalling that countries are in every period distributed along the segment I with the distribution G, financial market clearing can be formalized as follows:

$$\int_{i \in I} q\left(B'^i, s^i\right) B'^i G\left(di\right) = 0, \qquad (8)$$

where B'^i is the asset demand of country $i \in I$ in the current period. We focus on the steady state equilibrium where the price of the safe asset q is constant. We can provide a more formal definition of the equilibrium.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a price q, risk premia $(\delta(B,s))_{s=u,p,B\in\mathbb{R}}$ and policy functions $(B'(B,s))_{s=u,p,B\in\mathbb{R}}$, $l(B,p)_{B\in\mathbb{R}}$ and the level of production in autarky l^d such that: 1) for a given price q and given risk premia $(\delta(B,s))_{s=u,p,B\in\mathbb{R}}$, policy functions solve the program (2)–(6); 2) the risk premia are consistent with perfect competition and full diversification for financial intermediaries; 3) the price q is such that the financial*

market clears, i.e., that equation (8) holds.

3   Market equilibrium

We can present the main properties of the market equilibrium. We start with the next proposition, which characterizes the equilibrium price schedule, and which is standard in this type of environment.

Proposition 1 (Price schedule) *There exists two asset thresholds $\overline{B}^e$ and $\overline{B}^u$, $\overline{B}^e < \overline{B}^u < 0$ such that:*

$$q\left(B', p\right) = \begin{cases} q & \text{if } B' \geq \overline{B}^u, \\ \alpha q & \text{if } \overline{B}^e \leq B' < \overline{B}^u, \\ 0 & \text{if } B' < \overline{B}^e, \end{cases} \quad \text{and } q\left(B', u\right) = \begin{cases} q & \text{if } B' \geq \overline{B}^u, \\ (1-\rho)q & \text{if } \overline{B}^e \leq B' < \overline{B}^u, \\ 0 & \text{if } B' < \overline{B}^e. \end{cases}$$

Proposition 1 states that the price schedule is a step function with two thresholds $\overline{B}^e$ and $\overline{B}^u$, where it can be shown that $\overline{B}^e < \overline{B}^u$. If a country borrows less than the smaller threshold $\overline{B}^u$, then it will not default in the next period, no matter its idiosyncratic shock. In this case, the price of its claim is the price of a safe asset q. If the country borrows strictly more than $\overline{B}^u$ but less than a second threshold $\overline{B}^e$, it will default in case of a bad idiosyncratic shock, but it will not do so in case of a good idiosyncratic shock. As a consequence, the price of the claims is the price of the safe claim corrected by the probability to be productive in the next period, which is $1 - \rho$ for unproductive countries, and α for productive ones. Finally, for borrowed amounts that are strictly higher than $\overline{B}^e$, the country defaults no matter the next-period idiosyncratic shock, so that the price of the claim is 0. An implication of Proposition 1 is that productive countries never default. Indeed, if a country borrows a positive amount which induces a default when productive in the next period, the country will then always default in the next period, implying that the price of its claim is 0, which is a contradiction as the country cannot borrow in that case.

3.1   Euler equations and budget constraints

Quasi-linearity in the labor supply considerably simplifies the analysis of the equilibrium. To begin with, the first-order condition for the labor supply l of productive countries in (5) yields

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$u'(c_0) = 1$, where c_0 is the consumption of productive countries. In consequence, we obtain that the consumption level is $c_0 = u'^{-1}(1)$. Using the budget constraint (1), the value function (5) becomes:

$$V^c(B, p) = V^c(0, p) + B. \tag{9}$$

In consequence, the value function of productive countries participating in financial markets is linear in the beginning-of-period asset endowment. All productive non-defaulting countries have therefore the same marginal utility for consumption, which is independent of their beginning-of-period endowment.

We now prove that the equilibrium is characterized by a sequence $\{B_k\}_{k=0,\dots,\infty}$, such that all productive countries participating in financial markets save B_0 and participating countries who are unproductive for $k \geq 1$ consecutive periods save the same amount B_k. In equilibrium, the saving (or borrowing) decision of agents thus simply depends on their idiosyncratic production history, which considerably simplifies aggregation.

We adopt a guess-and-verify strategy to simplify the exposition. Assume that countries default after being unproductive for $D \leq \infty$ consecutive periods, where the case $D = \infty$ corresponds to countries never defaulting. The first-order conditions for asset choices are:

$$q = \beta\left(\alpha + (1-\alpha)u'(c_1)\right), \tag{10}$$

$$qu'(c_k) = \beta\left(1 - \rho + \rho u'(c_{k+1})\right), \quad k = 1, \dots, D-2, \tag{11}$$

$$qu'(c_{D-1}) \geq \beta, \tag{12}$$

where c_k, $k = 1, \dots, D-1$ denotes the consumption level of a participating country who is unproductive for the k consecutive previous periods. The budget constraints (1) can now be expressed as:

$$B_{k-1} = c_k - \bar{l} + qB_k, \ k = 1, \dots, D-2, \tag{13}$$

$$B_{D-2} = c_{D-1} - \bar{l} + q(1-\rho)B_{D-1}, \tag{14}$$

We denote by l_{k0} the labor supply of productive countries who have been unproductive for the k previous periods and just turned productive. By extension, l_{00} denotes the labor supply of productive countries who were already productive in the previous period, and l_{-10} is the labor supply of participating productive countries who were excluded from financial markets in

the previous period and who just reentered financial markets. For the sake of simplicity, the beginning-of-period wealth of these countries will also be assumed to be null: $B_{-1} = 0$. The budget constraint (1) implies:

$$l_{k0} = u'^{-1}(1) - B_k + qB_0, \ k = -1, 0, \ldots, D - 1, \tag{15}$$

The intuition for such a simple equilibrium structure, is that all productive countries choose the same consumption level $u'^{-1}(1)$ and the same asset holding B_0 due to the quasi-linearity of the utility function (and the linearity of the value function). The labor supply of productive countries l_{k0} adjusts to compensate for the difference in their beginning-of-period wealth levels. This wealth level depends on country's productive status in the previous period and in particular on the number k of consecutive past periods of being unproductive (with $k = 0$ corresponding to a productive status and $k = -1$ to a default status).

Remark 1 (Notation) *To lighten notation, we will consistently use below the notation that we introduced in this section: For a variable x, x_k is the quantity associated to a country unproductive for $k = 1, \ldots, D - 1$ periods, while x_{k0} denotes the quantity for a productive country unproductive for the $k = -1, 0, \ldots, D - 1$ previous periods. For instance, x_{20} concerns a productive country who was unproductive in the last two previous periods. The quantity associated to productive country who was already productive in the previous period is x_{00}, while x_{-10} is for a productive country who was excluded. When the past status does not matter for a productive country –for instance in the case of consumption level–, we simply use the notation x_0.*

3.2 Default conditions and limited-heterogeneity equilibrium.

Countries will default after being unproductive for exactly D consecutive periods if the following conditions hold: (i) paying back debt dominates the default option if countries unproductive for less that $D - 1$ consecutive periods; (ii) unproductive countries for exactly D consecutive periods would be worse off paying back their debt than defaulting; (iii) productive countries who have been previously unproductive for k consecutive periods choose to pay back their debt contracted

while unproductive and do not default. These conditions can be formalized as follows:

$$V_u^c(B_k) \geq V_u^d, \ 0 < k < D, \tag{16}$$

$$V_p^c(B_k) \geq V_p^d, \ 0 < k < D. \tag{17}$$

We can now express the financial market clearing condition. First, denote n_k the population of participating countries being unproductive for k consecutive periods for $1 \leq k \leq D$. From transition probabilities, we have that $n_k = \rho^k n_0$ for $k = 1, \ldots, D-1$, where n_0 denotes the mass of productive countries participating in financial markets. The explicit expression of these population share is relegated to the Appendix A to save some space. Using these notations and substituting for equilibrium population shares and prices, the financial market equilibrium (8) can be expressed as follows:

$$\sum_{k=0}^{D-2} n_k B_k + (1-\rho)n_{D-1}B_{D-1} = 0. \tag{18}$$

To summarize, the equilibrium is is thus characterized by: 1) a number of periods D before defaults, with possibly $D = \infty$, if default never occurs; 2) consumption levels $(c_k)_{k=0,\ldots,D-1}$, asset demands $(B_k)_{k=0,\ldots,D-1}$ and labor efforts $(l_{k0})_{k=-1,0,\ldots,D-1}$ satisfying Euler equations (10)–(12), budget constraints (13)–(15), while conditions (16)–(17) hold; 2) a price for the safe asset q, enabling the market to clear and equation (18) to hold.

As we show in our quantitative exercise of Section 6, the value D is finite for standard calibrations. The economy is then populated by a finite number of different types of countries, where each type chooses the same consumption and wealth. More precisely, $D-1$ unproductive types participating in financial markets are characterized by the number $k = 1, \ldots, D-1$ of consecutive periods for which they are unproductive. Furthermore, $D+1$ productive types participate in financial markets and are characterized by their status in the previous period, which could either be: (i) unproductive, (ii) productive, or (iii) excluded from financial markets. As explained above, productive countries choose the same consumption and asset holding and only differ with respect to their labor effort. In addition to these $2D$ participating types, 2 additional types of countries are excluded from financial markets, and differ according to their productive status. On the whole, the economy is populated by $2D+2$ different types of countries.

We can now compute the welfare for each type of country. Consistently with our previous

notation, we denote V_k the intertemporal welfare of a country participating in financial markets and unproductive for $k = 1, \ldots, D - 1$ periods, while V_{k0} denotes the intertemporal welfare of a productive country participating in financial markets, which was productive in the $k = -1, \ldots, D - 1$ previous periods. Formally, we have:

$$V_{k0} = u\left(u'^{-1}(1)\right) - l_{k0} + \beta\left(\alpha V_{00} + (1 - \alpha)V_1\right), \ k = -1, \ldots, D - 1, \tag{19}$$

$$V_k = u(c_k) - \bar{l} + \beta\left((1 - \rho)V_{k0} + \rho V_{k+1}\right), \ k = 1, \ldots, D - 1. \tag{20}$$

Using expressions (19) and (20), participation conditions (16) and (17) become $V_k \geq V_u^d$ and $V_{k0} \geq V_p^d$ for $k = 1, \ldots, D - 1$.

The next proposition further characterizes the equilibrium structure.

Proposition 2 (Limited-heterogeneity equilibrium characterization) *The price q veri-fies $q \geq \beta$ and we have* either*:*

(a) $D = \infty$, $q > \beta$ *and* $\lim_{k \to \infty} B_k = -\frac{\bar{l}}{1-q}$*; or*

(b) $D < \infty$ *and* $V_p^c(B_{D-1}) = V_p^d$.

Proposition 2 states that two types of equilibrium can exist. The first one (a) is an equilibrium, in which default never occurs. The second one (b) is an equilibrium, in which the borrowing limit B_{D-1} for unproductive countries is determined by their default incentive when becoming productive next period. In this equilibrium, unproductive countries borrow more and more as they stay unproductive, until they reach an amount for which they would default in the following period in case they become productive. Indeed, a productive country, who starts the period with a large amount of debt would have to provide a significant labor effort to repay its debt and it may thus be better off defaulting.

Although characterizing possible equilibrium structure, Proposition 2 is silent about the possible equilibrium uniqueness. Section 3.3 below proves that multiple equilibria are possible.

Finally, we will compare the multiple equilibria using an utilitarian welfare criterion. Other criterion could be used, but this one seems to be the most natural in heterogeneous-agent economies, for reasons already discussed in Aiyagari (1994), and is thus widely used in the literature. Using the share of different types of countries, the expression of aggregate welfare

W^a is simply:

$$W^a = \sum_{k=-1}^{D-1} n_{k0}V_{k0} + \sum_{k=1}^{D-1} n_k V_k + n_p^d V_p^d + n_u^d V_u^d. \tag{21}$$

Aggregate welfare is the sum of the intertemporal welfare of countries participating in financial markets, being both productive and unproductive, and the intertemporal welfare of countries excluded from financial markets, who are both productive and unproductive.

3.3 Multiple equilibria

The previous section has characterized the equilibrium structure in the general case. We now explain why non-trivial multiple equilibria can occur, with different numbers of periods before default (i.e., different values of D), different amounts of liquidity, and different prices for the safe asset (and thus different values for the real interest rate).

Countries self-insure themselves in good times by purchasing the safe asset. The quantity of safe assets amounts to the risk-adjusted quantity of debt in the economy. When the safe asset is scarce, its price q is high and the risk-free interest rate r is low. Self-insurance is expensive as the return on savings is low, and countries have low incentives to save in good times for self-insurance motives. In consequence, the demand of countries for the safe asset is low and consumption smoothing is poor, as it can be observed from first-order conditions of participating agents (10). Participating in financial markets is therefore not very attractive since risk sharing and consumption smoothing benefits are small. In these conditions, the opportunity cost to default is low, and the number of defaults in equilibrium is high, implying a small D in equilibrium. The quantity of "liquidity" available either for borrowing or lending is low. This is in turns consistent with the fact that liquidity is scarce and the price q of the safe asset is high.

On the opposite, when liquidity is abundant, the welfare gains of self-insurance are high, while the incentives to default are low. In equilibrium, countries default less and D is high in equilibrium.

To our knowledge, this identification of the possibility of equilibrium multiplicity in models with incomplete markets and default is new in the literature. The key mechanism is the endogeneity of the price of the safe assets. As a consequence, such multiple equilibria cannot occur in environments where the price of the safe asset is exogenous. Indeed, with exogenous prices, Auclert and Ronglie (2016) show that the equilibrium is unique under some general conditions,

that are fulfilled here.

Our main proposition in this section is the ability to rank these multiple equilibria according to our welfare criterion.

Proposition 3 (Ranking of multiple equilibria) *Multiple equilibria can be ranked according to their aggregate welfare. The higher the number D of consecutive periods before default, the higher the aggregate welfare. As a consequence, the competitive equilibrium that maximizes the aggregate welfare is the one with the largest admissible D.*

The proof can be found in Appendix. Intuitively, the economy with a lower amount of default (and a higher D) is characterized by a higher quantity of liquidity and therefore more risk sharing, which is welfare improving.

4 Identifying distortions in the market economy: The constrained-efficient allocation

We now investigate the distortions in the market equilibrium corresponding to the highest welfare and the largest D. We prove that this equilibrium is not constrained-efficient. Indeed, even though this equilibrium is the market equilibrium featuring the highest quantity of liquidity, the liquidity remains insufficient for constraint-efficiency, because countries do not completely internalize the social costs of default.

To analyze these distortions more formally, we solve for the allocation of a quasi-planner facing the same risk structure as in the market economy, but who cannot transfer resources across countries and who cannot prevent countries from defaulting.[7] The quasi-planner decides the consumption level, the saving amount and the labor supply of all countries, while internalizing the financial market clearing condition. Our approach is similar to the one of Davila, Hong, Krusell, and Rios-Rull (2012), who define a notion of constrained-efficiency in an incomplete insurance-market environment without default. As a consequence, the individual budget constraint of each country must be fulfilled. In addition, since the quasi-planner cannot forbid

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[7]The term quasi-planner refers to a planner maximizing social welfare with country-specific budget constraints as, for instance, in Veracierto (2008) in a different environment. A general theory for the construction of a quasi-planner economy to identify distortions in incomplete-insurance market environment is presented in LeGrand and Ragot (2017). We apply it here to economies with default.

default, participation constraints of countries must also hold. The optimal allocation of the quasi-planner defines the constrained-efficient allocation, when the pecuniary externality is internalized. We show in Section 5 how this constrained-efficient allocation can be implemented by a simple market mechanism.

We now turn to the formal program of the quasi-planner. We use the same notations as in the market economy of Section 2, except that we add a tilde for the quasi-planner allocation. First, the planner decides upon the optimally the default decision $\tilde{D}$. The quasi-planner also chooses the consumption $\tilde{c}_0$, the net savings and $\tilde{B}_0$, and the labor supply $\tilde{l}_{k0}$ of productive countries who have been unproductive for $k = -1, 0, \ldots, \tilde{D}$ consecutive periods. The planner also chooses the consumption $\tilde{c}_k$ and the savings $\tilde{B}_k$ of unproductive countries for $k = 1, 2, \ldots, \tilde{D}$ consecutive periods. The net wealth of countries re-entering financial markets is set to zero, as in the market economy: $\tilde{B}_{-1} = 0$. The quasi-planner cares about the instantaneous aggregate utility $\tilde{U}$ expressed as

$$
\tilde{U} = \sum_{k=-1}^{\tilde{D}-1} \tilde{n}_{k0} \left(u\left(\tilde{c}_0\right) - \tilde{l}_{k0} \right) + \sum_{k=1}^{\tilde{D}-1} \tilde{n}_k \left(u\left(\tilde{c}_k\right) - \bar{l} \right) \tag{22}
$$
$$
+ \tilde{n}_p^d \left(u\left(\tilde{c}_p^d\right) - \tilde{l}_p^d \right) + \tilde{n}_u^d \left(u\left(\tilde{c}_u^d\right) - \bar{l} \right),
$$

where the tilda in population shares highlights their dependence in $\tilde{D}$. Formal expressions of population shares are the same as in the market economy, given in Appendix A. The program of the quasi-planner can be expressed recursively as follows

$$
\tilde{W} = \max_{\substack{\{\tilde{q}, \tilde{D}, \tilde{c}_p^d, \tilde{l}_p^d, \tilde{c}_u^d\}, \\ \{\tilde{l}_{k0}\}_{-1 \le k < \tilde{D}-1}, \{\tilde{c}_k\}_{1 \le k < \tilde{D}-1}, \{\tilde{B}_k\}_{0 \le k < \tilde{D}-1}}} \tilde{U} + \beta \tilde{W}', \tag{23}
$$

subject to the set of country budget constraints, which are similar to (13)–(15) in the market economy:

$$
\tilde{B}_{k-1} = \tilde{c}_k - \bar{l} + \tilde{q}\tilde{B}_k, \quad 1 \le k < \tilde{D} - 1, \tag{24}
$$
$$
\tilde{B}_{\tilde{D}-2} = \tilde{c}_{D-1} - \bar{l} + \tilde{q}\left(1 - \rho\right)\tilde{B}_{\tilde{D}-1}, \tag{25}
$$
$$
\tilde{c}_0 + \tilde{q}\tilde{B}_0 = \tilde{l}_{k0} + \tilde{B}_k, \quad -1 \le k \le \tilde{D} - 1, \tag{26}
$$
$$
\tilde{c}_p^d = \varphi\tilde{l}_p^d, \text{ and } \tilde{c}_u^d = \varphi\bar{l}, \tag{27}
$$

17

and subject to individual participation constraints (similar to (16) and (17) in the market economy):

$$\tilde{V}_{k0} \geq \tilde{V}_p^d, \quad -1 \leq k < D, \tag{28}$$

$$\tilde{V}_k \geq \tilde{V}_u^d, \quad 1 \leq k < D. \tag{29}$$

The quantities $\tilde{V}_{k0}$ and $\tilde{V}_k$ represent the intertemporal welfare of different country types, as in (19) and (20) for the market economy. The last constraint of the quasi-planner program is the financial market clearing:

$$\sum_{k=0}^{\tilde{D}-1} \tilde{n}_k \tilde{B}_k = 0. \tag{30}$$

The key-difference between the market economy and the constrained-efficient allocation, is that the quasi-planner internalizes the effect of saving and default decisions on the price of the safe asset, and thus on aggregate risk sharing.

The solution of the quasi-planner program is straightforward. We present here the corresponding first-order conditions to provide intuitions about the constrained-efficient allocation. We denote as λ the Lagrange multiplier of the financial market clearing constraint (30), which corresponds to the social value for the planner of one additional unit of world safe asset. Euler equations can be written as:

$$\tilde{q} + \lambda = \beta \left(\alpha + (1 - \alpha) u' (\tilde{c}_1) \right), \tag{31}$$

$$\tilde{q} u' (\tilde{c}_k) + \lambda = \beta \left(1 - \rho + \rho u' (\tilde{c}_{k+1}) \right), \quad 1 \leq k \leq \tilde{D} - 2, \tag{32}$$

$$q \tilde{u}' \left(c_{\tilde{D}-1} \right) + \lambda \geq \beta. \tag{33}$$

Note that full risk sharing among participating countries is a solution of the quasi-planner program. Indeed, a solution to the set of equations (31)–(33) is $\tilde{q} + \lambda = \beta$ and $u' (\tilde{c}_k) = 1$ for $k = 0, \ldots, \tilde{D} - 1$. The quasi-planner distorts the saving choices of countries, so as to reach full risk sharing for countries participating in financial markets. This fosters the intertemporal welfare of participating countries, and therefore decreases the default incentives. This maximizes the quantity of the safe asset and aggregate welfare. The equilibrium value of $\tilde{D}$ is pinned down by the participation constraints (28) and (29). Indeed, full risk sharing is obtained by countries accumulating debt when unproductive and continue to do so until they would be better off

defaulting instead of paying back the debt when becoming productive. The price of the safe asset is then determined by the world financial market equilibrium together with all budget constraints. The next proposition summarizes this quasi-planner allocation.

Proposition 4 (Constrained-efficient allocation) *The constrained-efficient allocation is characterized by $\tilde{c}_k = u'^{-1}(1)$ for $k = 1, \ldots, \tilde{D} - 1$.*

In general, the constrained-efficient allocation differs from the market allocation characterized in Section 3.1. To see this observe that $\tilde{c}_k = u'^{-1}(1)$ in the market economy implies $q = \beta$. However, in this case, the saving decisions B_k, $k \le \tilde{D} - 1$ have no reason to satisfy the financial market clearing. Since the quasi-planner internalizes the financial market clearing condition, this distorts Euler equations, and enables the quasi-planner to reach perfect risk sharing without the constraint of individual Euler equations.

This remark provides an intuition for the decentralization scheme presented in the next section. Indeed, perfect risk sharing implies an increase in the liquidity available for self-insurance in the market economy, but without distorting individual Euler equations. The decentralization scheme that we propose involves an international fund issuing liquidity and allowing countries to perfectly share risk.

5 A decentralization device: An international liquidity fund

We now show that an international liquidity fund can help to restore constrained-efficiency. The environment is similar to the market economy described in Section 2 (we keep the same notations), except the additional presence of the fund, issuing one-period risk-free debt financed by the contributions of member states.[8]

We assume that the contributions and the participation in the fund are regulated by the following rules.

1. The fund cannot implement contributions, conditional on the productive status of member countries.

---

[8]The risk-free asset issued by the funds has some obvious connections with the Special Drawing Rights (SDRs) of the IMF. We discuss the differences below.

2. The fund can choose a different contribution for countries joining the fund in the current period.

3. Countries can freely choose to belong to the fund, and can always choose to opt out.

4. Contributing countries commit not to transact with countries not belonging to the fund.

Let us comment the above rules. First, we rule out contributions that can be contingent on the productive status and for instance be different for productive and unproductive countries. The fund could otherwise increase welfare using differentiated contributions, thereby providing partial insurance against the productive risk. We exclude this obvious comparative advantage for the fund, which could thus justify the existence of the fund. Instead, we assume that the fund is only able to discriminate countries just joining the fund from countries belonging to the fund for several periods (Assumption 2). This realistic weaker assumption is enough to exactly reproduce the constrained efficient aggregate welfare. Third, countries can decide to participate in the fund or not. The participation decision is sovereign. Fourth, contributing countries are prevented from transacting with non-contributing countries. Otherwise, countries would have no incentive to pay their contribution to the fund, as they could benefit from the world liquidity (and paying the world interest rate on any safe asset by absence of arbitrage) without paying contributing costs. No country would contribute to the fund.

We now turn to the formalization of the fund. We denote as F' the debt issued by the fund at the price q in a given period, while F is the amount the fund repays. Note that the fund pays the world risk-free interest rate, by absence of arbitrage. The total measure of countries participating in the fund is denoted m_p, while m_{-10} is the measure of countries just joining the fund in the current period. These quantities can differ from proportions n_p and n_{-10}. Countries joining the fund receive a transfer subsidy amounting denoted τ_{-10}, and other contributing countries pay τ. The notation τ_{-10} refers to a country previously in the default state (state -1) becoming productive, re-entering financial markets and joining the fund with a zero wealth (state 0). With these notations, the fund budget constraint is:

$$F + m_{-10}\tau_{-10} = qF' + (m_p - m_{-10})\tau$$

The word financial market clearing condition can now be expressed as:

$$\sum_{k=0}^{D-1} n_k B_k = F'. \tag{34}$$

Finally, budget constraints of countries are the same as those in absence of fund in equations (13)–(15), except that countries additionally receive a transfer if they decide to join the fund. This contribution (or the subsidy) amounts to either τ_{-10} or τ, depending on whether the concerned country just joins the fund or not. The proposition summarizes the main result about the fund-economy.

Proposition 5 (Equilibria in the family-head and decentralized economies) *There exist values for τ, τ_{-10}, and F, such that the aggregate welfare in the fund-economy is identical to the one in a constrained-efficient economy. In the fund-economy, all countries either participate in the fund or are in autarky.*

The proof can be found in Section D of Appendix. The equilibrium quantity F is determined to guarantee that the asset price is $q = \beta$. The transfers τ and τ_{-10} are set first to balance the fund budget, and second to implement the same default decision as in the constrained-efficient economy. At the equilibrium of the fund-economy, all countries participating to financial markets contribute to the fund.

It is noteworthy that Proposition 5 proves the existence of transfers τ, and τ_{-10}, and of a fund amount F for the aggregate welfare in the fund economy to be identical to the one in a constrained-efficient economy. However, the proposition is silent about uniqueness. Indeed, an equilibrium in which no country participates in the fund always exists. In that case, an atomistic country has no interest in contributing. The fund becomes meaningless and the economy is identical to the market economy.

This fund is obviously reminiscent of the IMF issuing Special Drawing Rights, which are international store of values. The SDR is a reserve asset issued by the IMF since 1970, with the explicit goal to reduce the world shortage of liquidity (see Williamson 2009 for a summary of the history of the introduction of SDRs). The main difference between the fund introduced in this section and the SDRs is that the interest on assets issued by the fund is a yearly interest rate, whereas the remuneration of SDRs is an average of short-run (3 months) interest rates on

a basket of currencies.[9] The next section provides a quantification of the optimal amount of such a world liquidity.

6   A quantification of the size of the fund

We now provide a quantitative exercise to identify the distortions in the market economy and to quantify the size of the fund allowing to reach the constrained efficient allocation. To discipline the model we use both the literature on sovereign default, and the literature on disaster events, which is consistent with the two-state Markov structure we use for the idiosyncratic risk. The period is a year. The model has seven parameters, for which seven targets or values are provided.

The curvature of the utility function is set to $\sigma = 2$, as in Arellano (2008), and the discount factor is set to $\beta = .96$ to obtain an annual world interest rate around of 4% in the world-market economy.

We set the probability of staying productive to 96.5% such that the probability to switch from productive to unproductive is 3.5%. This is equal to the annual probability to enter into a disaster state, as found by Barro and Ursua (2011). The probability to stay unproductive is set to 0.87% such that the average number of periods in a unproductive state amounts to 7.7 years. This value is close to the average duration of a disaster state, approximately equal to 7 years. The production when unproductive is set to $\bar{l} = 0.99$, which implies a cumulative output loss before default of 7%. This is consistent with Tomz and Wright (2013), who report that output is 8% below trend when default occurs. The probability to reenter the economy is set to $\theta = 15\%$, which generates an average length of financial market exclusion of 7 years, which roughly matches the 6.5 years reported by Tomz and Wright (2013). Finally the cost of default is set to 1.5% of output ($\varphi = 0.985$). This value is chosen to trigger default after 7 consecutive periods in the unproductive state (as one possible equilibrium, see below), which is the average length for a disaster. This value implies that 16.4% of all countries are in default, which is close to the empirical value of 19% found by Tomz and Wright (2013). An additional outcome of this calibration is that the unconditional probability of default amounts to 1.0%. This value is slightly below standard empirical estimates, which vary between 1.8% and 2.2%. Table 1 summarizes the parameter values and targets.

---

[9]The weekly interest rates on SDRs is provided at `http://www.imf.org/external/np/fin/data/sdr_ir.aspx`,

Parameters	Values	Model outcomes and references
Discount factor	$\beta = 0.96$	Interest rate around 4%
Utility function	$\sigma = 2$	Arellano (2008)
Persistence of good state	$\alpha = 96.5\%$	3.5% disaster probability (Barro and Ursua 2011)
Persistence of disaster	$\rho = 87\%$	Average duration of 7.7 years (Barro and Ursua 2011)
Labor in disaster state	$\bar{l} = 0.99$	7% cumulative GDP fall when default (Tomz and Wright 2013)
Prob. to re-enter	$\theta = 15\%$	7 years of exclusion after default (Tomz and Wright 2013)
Default cost	$\varphi = 0.985$	Default after 7 years in unproductive state (Barro and Ursua 2011)

Table 1: Calibration

6.1 Market equilibria

The first outcome of the exercise is that multiple equilibria are pervasive. The previous calibration is consistent with an equilibrium where $D = 7$, i.e. where default occurs after 7 years in the unproductive states, but also with a range of equilibria with D varying from $D = 2$ to $D = 7$ (there is no equilibrium for $D > 7$). As was explained in the previous section, the source of equilibrium multiplicity is the endogeneity of the benefit to participate in financial markets, which depends on the default decision. In Table 2, we report for all equilibria some key statistics: interest rate, the fraction of countries in default and the unconditional default probability.

D	2	3	4	5	6	7
Interest rate (%)	4.16	4.13	4.13	4.13	4.13	4.13
Share of countries in default (%)	30.4	27.0	23.9	21.1	18.6	16.4
Unconditional default probability (%)	2.1	1.82	1.6	1.3	1.2	1.0

Table 2: Equilibrium properties for various equilibria indexed by D

together with the explicit interest rate formula.

The equilibrium interest rate is lower in equilibrium where countries default less and it rapidly converges to the value of 4.13%. Note that the complete market interest rate is 4.17%. When D raises, both the supply and the demand for liquidity increases, as a smaller number of countries default and countries need to self-insure for a larger number of periods in unproductive state before defaulting. Consequently, the interest rate does not change a lot when D increases. In addition, and as can be expected, both the fraction of countries in default and the equilibrium default probability decrease with D.

As shown in Section 3.3, more risk sharing takes place in equilibria with larger D. We illustrate this aspect in Figure 1, where we plot the consumption profiles as a function of the number of consecutive unproductive priods, for different values of D. For sake of clarity, we only report $D = 2$, 4 and 7. When the country defaults and remains unproductive, its consumption is equal
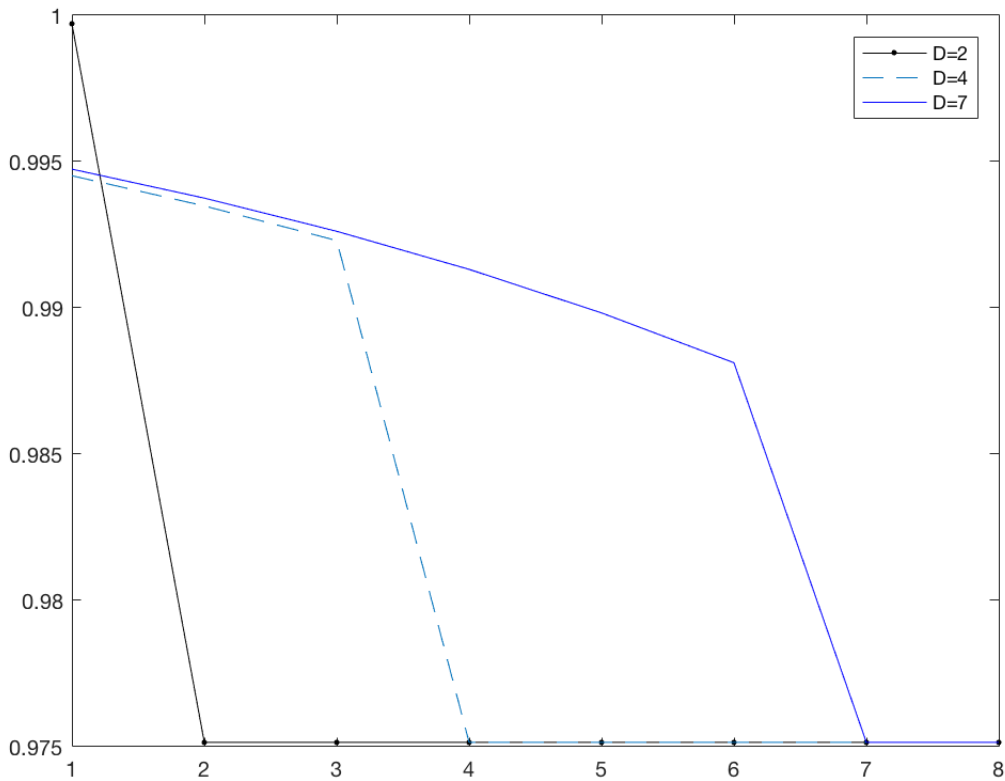

Figure 1: Consumption profiles as a function of the number of consecutive unproductive periods, for various D

to the autaraky value $\varphi \bar{l} = 0.9751$, which is therefore the lowest consumption value. In addition, the path of consumption is decreasing with the number of consecutive period in unproductive state (before default), for all equilibria under consideration. Consumption-smoothing is thus imperfect. The implications for welfare are presented in Section 6.3 below.

6.2   The equilibrium for the fund-economy

We now derive the equilibrium where the fund issues debt to replicate the constrained-efficient aggregate welfare. In the constrained-efficient equilibrium, we obtain $D^{opt} = 10$: countries default after being unproductive for 10 periods. The world interest rate is 4.17%, which is exactly the complete market interest rate $1/\beta - 1$. All countries participating in financial markets, either productive or not, enjoy the same consumption equal to 1. There is thus full risk sharing for countries participating in financial markets.

As shown in Section 5, this allocation can be implemented with three tools: a level of debt F, country-specific contributions τ, and a subsidy for countries joining the fund, τ_{-10}. With our calibration, implementing the constrained-efficient allocation implies the following fund characteristics. The fund must issue a debt level equal to 4.1% of world GDP. The average contributions of countries has to be equal to $\tau = 0.22\%$ of their own GDP. Finally, the subsidy by the fund of countries re-entering financial markets and joining the fund is $\tau_{-10} = 3.49\%$. This amount is necessary to compensate for the net discounted value of expected contribution to the fund.[10]

6.3   Welfare comparisons

We conclude our quantitative exercise by the comparison of the welfare properties of the multiple equilibria. For any equilibrium, we compute the aggregate welfare using the standard utilitarian criterion (see equation 21), which equally weights the intertemporal welfare of all countries. In Table 3, we report the aggregate welfare differences between market equilibria (for D varying from 2 to 7) and the fund economy where $D^{opt} = 10$. This aggregate welfare difference is expressed in consumption equivalent and corresponds to the relative decrease in consumption of each country, for the welfare in the fund economy to be identical to the one in the market

---

[10]Although relatively small, the size of the fund is much higher than the current amount of outstanding SDRs of the IMF which is less than 0.3% of world GDP in 2016.

economy for a given D.

D	2	3	4	5	6	7
Welfare difference (in consumption equivalent %)	-0.33	-0.27	-0.23	-0.18	-0.15	-0.11

Table 3: Welfare difference between market equilibria and the fund-economy in consumption equivalents.

As expected, the welfare in the fund-economy (where $D^{opt} = 10$) is always higher than the welfare in the market economy, for any possible equilibrium. The welfare cost decreases with D. In other words, the higher D, the higher aggregate welfare. This is consistent with Proposition 3, illustrating that the higher D, the stronger the risk sharing.

To conclude, it is noteworthy that welfare differences reported in Table 3 are sizable. For instance, the welfare difference between the market equilibrium for $D = 7$ and the fund-economy amounts to 0.11% in consumption equivalent. This welfare difference is approximately equal to two or three times the welfare gains of eliminating business cycles in standard representative-agent analysis. The reason for this large welfare variations comes from the disaster state (being unproductive for seven periods), which is much more persistent than bad aggregate shocks in typical business cycle analysis.

7   Conclusion

We provide a tractable model where sovereign default can be studied in general equilibrium. This model allows us to derive normative properties of the equilibrium structure implied by the endogenous quantity of the world safe asset. A salient property of the model is equilibrium multiplicity, that is due to an endogenous opportunity cost of default. In addition, all market equilibria exhibit too little risk sharing compared to a constrained efficient equilibrium. The additional result of the paper is that the constrained-efficient equilibrium can be attained by introducing a fund issuing safe asset based on the voluntary contribution of member states. The liability of the fund is around 4% of the world GDP. This quantity, though small, is still higher than the outstanding SDRs issued by the IMF.

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Appendix

A   Population shares

Using the model structure, one can deduce the following population shares

$$n_0 = \frac{\theta}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}} \frac{1-\rho}{2-\rho-\alpha}, \tag{35}$$

$$n_k = \rho^{k-1}(1-\alpha)n_0 \text{ for } k = 1, \ldots, D-1, \tag{36}$$

$$n_k = 0 \text{ for } k \geq D. \tag{37}$$

Denote n_u^d and n_p^d the mass of unproductive and productive agents who have defaulted and are excluded from financial markets:

$$n_p^d = \frac{(1-\theta)(1-\alpha)\rho^{D-1}}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}} \frac{1-\rho}{2-\rho-\alpha}, \tag{38}$$

$$n_u^d = \frac{(1-(1-\theta)\alpha)\rho^{D-1}}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}} \frac{1-\alpha}{2-\rho-\alpha}. \tag{39}$$

With our timing convention, the number of excluded countries re-entering financial markets after having being excluded is then $n_{-10} = \theta\left(\pi_{ee}n_p^d + \pi_{ue}n_u^d\right)$ in each period. It is then straightforward to deduce that the overall population (excluded and participating agents) of productive countries amounts to $\frac{\pi_{up}}{\pi_{up}+\pi_{pu}}$ and this of unproductive countries to $\frac{\pi_{pu}}{\pi_{up}+\pi_{pu}}$. Note that these populations are independent of the quantity D.

B   Proof of Proposition 2

The proof is in three steps.

B.1   Proof that: $q \geq \beta$.

Let us assume $q < \beta$. First, if $D < \infty$, equation (12), implies $u'(c_{D-1}) \geq \beta/q > 1$. We then deduce by induction from equation (11) that $u'(c_k) > 1$ for $k = 1, \ldots, D-1$. Equation (10) then implies that $q > \beta$, which is a contradiction. Second, if $D = \infty$, we first show from equations (10) and (11) that $1 > u'(c_k)$ for all k. Equation (11) also implies that $\rho(u'(c_{k+1}) - u'(c_k)) <$

$(1 - \rho)(u'(c_k) - 1) < 0$ and the sequence $(u'(c_{k+1}))_k$ is decreasing and positive. It thus admit a limit u'_∞, which verifies from (11), $u'_\infty \left(1 - \frac{\beta}{q}\right) = \frac{\beta}{q}(1 - \rho)(u'_\infty - 1)$. Since $q < \beta$, we must have $u'_\infty > 1$, which contradicts $1 > u'(c_k)$ for all k.

B.2 Case $D = \infty$

Assume that $q = \beta$. Using the Euler conditions of all countries (10) and (11), one can show recursively that $u'(c_k) = 1$ for all k. Hence, $c_k = u'^{-1}(1)$. Using the budget constraint one finds $u'^{-1}(1) + \beta B_{k+1} = \bar{l} + B_k$. It is then easy to show that $\lim_{k \to \infty} B_k = -\infty$ (any other case being impossible due to financial market clearing condition). This is a contradiction from Proposition 1 as the there is a threshold for debt in any equilibrium under consideration. In consequence $q > \beta$.

Since $q > \beta$, we obtain from Euler equations (10)–(12):

$$1 - \alpha < (1 - \alpha)u'(c_1)$$

$$u'(c_k) - 1 < \rho\left(u'(c_{k+1}) - 1\right), \quad k \geq 1,$$

which implies

$$u'(c_k) - 1 \geq \rho^{-k}\left(u'(c_1) - 1\right)$$

with $u'(c_1) > 1$. In consequence, $\lim_{k \to \infty} u'(c_k) = \infty$ and $\lim_{k \to \infty} c_k = 0$. Budget constraint (13) implies $\lim_{k \to \infty} B_k = -\frac{\bar{l}}{1-q}$.

B.3 Case $D < \infty$

Since the default occurs at date D, we have $V_k \geq V_u^d$ for $k = 1, \ldots, D-1$ and $V_{k0} \geq V_p^d$ for $k = -1, 0, \ldots, D-1$. Let assume that $V_{D-1,0} > V_p^d$. We have from equations (14) and (20):

$$V_{D-1} = u\left(B_{D-2} + \bar{l} - qB_{D-1}\right) - \bar{l} + \beta\left((1 - \rho)V_{D-1,0} + \rho V_u^d\right),$$

with $V_{D-1} \geq V_u^d$ and V_{D-1} decreasing in B_{D-1}. In consequence, the agent u unproductive for $D - 1$ periods could increase her welfare by decreasing slightly B_{D-1}. This would increase her welfare without affecting default incentives since $V_{D-1,0} > V_p^d$, which by continuity still holds after a small decrease in B_{D-1}.

C   Proof of Propositions 3

Let $D \geq 1$. Using equations (35)–(39) of Section A in Appendix, we can compute the impact of D on population shares.

- n_0 goes up with D:
$$\frac{\partial n^p}{\partial \rho^{D-1}} = -\frac{(1-\theta)(1-\alpha)n^p}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}} < 0.$$

- n_p^d goes down with D since the total population of p agents remains constant.

- $\sum_{k=1}^{D-1} n_k$ goes up with D:
$$\frac{\partial n^u}{\partial \rho^{D-1}} = -\frac{(1-\theta)(1-\alpha)n^p}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}} - \frac{\theta}{\theta + (1-\theta)(1-\alpha)\rho^{D-1}}\frac{1-\alpha}{2-\alpha-\rho} < 0.$$

- n_u^d goes down with D since the total population of u agents remains constant.

Consider the planner program that can rewritten as:

$$W = \max_{D \in \mathbb{N}_*} \quad \max_{\substack{\{c_p^{d,D}, l_p^{d,D}, c_u^{d,D}\}, \\ \{l_{k0}^D\}_{-1 \leq k < D}, \{c_k^D, B_k^D\}_{0 \leq k < D} \in \mathcal{A}_D}} U + \beta W',$$

where $\mathcal{A}_D$ is the feasible set for a given $D \in \mathbb{N}_*$. We have added superscripts D to allocations in order to highlight the dependence in D. We have

$$U = \sum_{k=-1}^{D-1} n_{k0}^D \left(u\left(c_0^D\right) - l_{k0}^D \right) + \sum_{k=1}^{D-1} n_k^D \left(u\left(c_k^D\right) - \bar{l} \right)$$
$$+ n_p^{d,D} \left(u\left(c_p^{d,D}\right) - l_p^d \right) + n_u^{d,D} \left(u\left(c_u^{d,D}\right) - \bar{l} \right)$$

where we also added the D superscript to population proportions. Let us show that $\mathcal{A}_D \subset \mathcal{A}_{D'}$ for any $D' \geq D$. To do so, we consider a feasible allocation $\left(c_p^{d,D}, l_p^{d,D}, c_u^{d,D}, \{l_k^D\}_{-1 \leq k < D}, \{c_{p,k}^D, B_k^D\}_{0 \leq k < D}\right) \in \mathcal{A}_D$. Using budget constraint (1), we can show that the following inequalities hold

$$B_{k-1}^{u,D} \geq c_k^{u,D} - \bar{l} + pB_k^{u,D}, 0 \leq k < D-1, \tag{40}$$

$$B_{D-2}^{u,D} \geq c_{D-1}^{u,D} - \bar{l} + p\left(1-\rho\right)B_{D-1}^{u,D}, \tag{41}$$

$$c_u^{d,D} \leq \varphi\bar{l}, \tag{42}$$

$$c_{p,k}^{p,D} + pB_0^{p,D} \leq l_k^{p,D} + B_k^{p,D}, 0 \leq k < D, \tag{43}$$

$$c_p^{d,D} \leq \varphi l_p^{d,D}. \tag{44}$$

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From the previous remark about the impact of D on population shares, we know that the population of p and u agents who are excluded is shrinking, while the population of p and u countries who participate is going up. We define for unproductive countries

$$
B_k^{D'} = \begin{cases} B_k^D & \text{for } n_k^D \text{ countries} \\ 0 & \text{for } n_k^{D'} - n_k^D \text{ countries} \end{cases},
$$

$$
c_k^{D'} = \begin{cases} c_{p,k}^D & \text{for } n_k^D \text{ countries} \\ c_{p,k}^d & \text{for } n_k^{D'} - n_k^D \text{ countries} \end{cases},
$$

$$
l_k^{D'} = \begin{cases} \bar{l} & \text{for } n_k^D \text{ countries} \\ \varphi \bar{l} & \text{for } n_k^{D'} - n_k^D \text{ countries} \end{cases},
$$

$$
c_u^{d,D'} = c_u^{d,D} \text{ for } n_u^{d,D'} \text{ countries,}
$$

$$
l_u^{d,D'} = \varphi \bar{l} \text{ for } n_u^{d,D'} \text{ countries,}
$$

where the idea is to "view" some excluded agents for D as participating agents for D', which is possible since we have (i) $n_k^D \leq n_k^{D'}$, (ii) $n_u^{d,D'} \leq n_u^{d,D}$, and (iii) $n_k^D + n_u^{d,D} = n_k^{D'} + n_u^{d,D'}$. It is obvious that constraint (42) holds for D', and that (40) and (41) also hold for D' (for a population of n_k^D agents that were in the same state for D). For the remaining population of $n_k^{D'} - n_k^D$ agents, equation (40) for D' becomes $c_u^{d,D'} - \bar{l} = c_u^{d,D} - \bar{l} + 0 = (\varphi - 1)\bar{l} \leq 0$, which therefore holds. By the same token, equation (41) for D' also holds.

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For productive agents, we similarly define

$$B_0^{D'} = \begin{cases} B_0^D & \text{for } n_0^D \text{ countries} \\ 0 & \text{for } n_0^{D'} - n_0^D \text{ countries} \end{cases},$$

$$c_0^{D'} = \begin{cases} c_0^D & \text{for } n_0^D \text{ countries} \\ c_p^{d,D} & \text{for } n_0^{D'} - n_0^D \text{ countries} \end{cases},$$

$$l_{k0}^{D'} = \begin{cases} l_{k0}^{p,D} & \text{for } n_{0k}^D \text{ countries} \\ \varphi l_p^{d,D} & \text{for } n_{0k}^{D'} - n_{0k}^D \text{ countries} \end{cases},$$

$$c_p^{d,D'} = c_p^{d,D} \text{ for } n_p^{d,D'} \text{ countries,}$$

$$l_p^{d,D'} = \varphi l_p^{d,D} \text{ for } n_p^{d,D'} \text{ countries,}$$

Note that equation (44) holds for D' by construction. Regarding, equation (43), we have $c_p^{d,D} \le \varphi l_p^{d,D} \le l_p^{d,D} \le l_p^{d,D'}$, which therefore holds.

In consequence the allocation $\left(c_p^{d,D}, l_p^{d,D}, c_u^{d,D}, \{l_k^D\}_{-1 \le k < D}, \{c_k^D, B_k^D\}_{0 \le k < D} \right)$ belongs to the feasible set $\mathcal{A}_{D'}$. We deduce that $\mathcal{A}_D \subset \mathcal{A}_{D'}$.

This has two consequences:

1. for a given price, feasible sets are increasing in D (for the inclusion). This implies that any welfare level for D can be reached for any $D' \ge D$. The welfare is therefore increasing in D.

2. by the same token, we can therefore rank competitive equilibria by an aggregate welfare criterion using D. This proves Lemma 3.

D   Proof of the proposition 5

To simplify the notations, we denote without variable and functions in the fund-economy with the same notations as in Sections 2 and 3. For instance, remind that a variable x related to a country re-entering financial markets is denoted x_{-10}. The program of such a country is

$$V_{-10} = u(c_{-10}) - l_{-10} + \beta \left(\alpha V_{00}(B_0) + (1 - \alpha)V_1(B_0) \right),$$

$$\text{s.t. } c_{-10} = l_{-10} + \tau_{-10} - qB_0.$$

The budget constraint of the fund is

$$(m_p - m_{-10})\tau + qF = F + m_{-10}\tau_{-10}, \tag{45}$$

while the financial market clearing can be expressed as

$$\sum_{k=0}^{D-1} n_k B_k = F. \tag{46}$$

An equilibrium for the fund economy is a default horizon D, a fund policy (F, τ_{-10}, τ), an asset price q, allocations $c_p^d, l_p^d, c_u^d, \{l_k\}_{-1 \leq k < D}, \{c_k, B_k\}_{0 \leq k < D}$, such that 1) for a given price q and given fund policy (F, τ_{-10}, τ), allocations are consistent with individual country programs; 2) the price q is such that the financial market clears, i.e., that equation (46) holds; 3) the fund policy is balanced : Equation (45) holds. Note that in this equilibrium, it is straghtforward to show that Proposition 2 holds, as the proof doesn't depend on contributions being paid or not by countries.

We start proving the following lemma.

Lemma 1 (existence) *There exists (F, τ, τ_{-10}), such that the fund economy equilibrium is characterized by: $D = \tilde{D}$, $q = \beta$, and all countries participating in financial markets participate in the fund.*

We show how to find (F, τ, τ_{-10}) satisfying the following conditions:

$$m_p = n_p \text{ and } m_{-10} = n_{-10}, \tag{47}$$

$$q = \beta, \tag{48}$$

$$F = \frac{n_p - n_{-10}}{1 - \beta}\tau - \frac{n_{-10}}{1 - \beta}\tau_{-10}, \tag{49}$$

$$V_{-10}(0) = \tilde{V}_{-10}(0). \tag{50}$$

Condition (47) guarantees that all countries participating in financial markets participate in the fund. Condition (48) ensures that the price is the optimal one, such that the Euler equations are not distorted. Condition (49) ensures that the budget of the fund is balanced. Finally, (50) guarantees that $D = \tilde{D}$, as shown below.

Proof of lemma 1

The proof consists in proving that finding (F, τ, τ_{-10}) such that equations (47)–(50) hold in the fund economy equilibrium boils down to solving a linear. We follow a guess and verify strategy. assume that equations (47)–(50) hold and that $D = \tilde{D}$ and $q = \beta$. We derive four linear conditions defining $(B_0, F, \tau, \tau_{-10})$. Then for these values of (F, τ, τ_{-10}), the fund market economy equilibrium must satisfy $D = \tilde{D}$ and $q = \beta$.

First equation

As $V_{-10}(0) = \tilde{V}_{-10}(0)$, the welfare in the fund-economy when reentering the economy is the same as in the constrained-efficient economy. It is easy to see that this implies that the autarky welfare levels in both economies are the same: $V_p^d = \tilde{V}_p^d$ and $V_u^d = \tilde{V}_u^d$.

Using results of Proposition 2, we must have in the market economies $V_p^d = V_c^p(B_{D-1})$. This yields a first constraint on (F, τ, τ_{-10}).

Indeed, as $D = \tilde{D}$ and $q = \beta$. The first order conditions, that are identical to (10)–(12), for $k = 0, \ldots, D - 1$ yield

$$c_0 = c_k = u'^{-1}(1) \equiv c.$$

Any participating country is fully insured against the productivity risk. The budget constraints, similar to (24) and (25), of unproductive countries except that they include the payment of the fund fee τ implies

$$B_{k+1} = \frac{\bar{l} - \tau - c}{\beta} + \frac{B_k}{\beta},$$

or after backward iteration:

$$B_k = \frac{B_0}{\beta^k} + (c - \bar{l} + \tau)\frac{1 - \beta^{-k}}{1 - \beta}. \tag{51}$$

Then welfare of participating countries that are unproductive for $k \leq \tilde{D} - 1$ periods is

$$V_k = u(c) - \bar{l} + \beta\left((1 - \rho)V_{k0}(B_k) + \rho V_{k+1}\right),$$
$$= u(c) - \bar{l} + \beta(1 - \rho)V_p^c(0) + \beta(1 - \rho)B_k + \beta\rho V_{k+1},$$

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Solving forward yields:

$$V_1 = \left(u(c) - \bar{l} + \beta(1-\rho)V_p^c(0)\right) \frac{1 - (\beta\rho)^{D-1}}{1 - \beta\rho} \tag{52}$$

$$+ (c - \bar{l} + \tau)\frac{\beta(1-\rho)}{1-\beta} \frac{1 - (\beta\rho)^{D-1}}{1 - \beta\rho}$$

$$+ \left(B_0 - \frac{c - \bar{l} + \tau}{1 - \beta}\right)(1 - \rho^{D-1}) + (\beta\rho)^{D-1} V_u^d.$$

The welfare of a productive participating country with a wealth 0 in the fund economy is

$$V_p^c(0) = u(c) - (c + \tau + \beta B_0) + \beta\alpha V_p^c(0) + \beta\alpha B_0 + \beta(1-\alpha)V_1,$$

or after the expression (52) of V_1:

$$(1 - \beta\alpha)\, V_p^c(0) = u(c) - (c + \tau + \beta B_0) + \beta\alpha B_0$$

$$+ \left(u(c) - \bar{l} + \beta(1-\rho)V_p^c(0)\right)\beta(1-\alpha)\frac{1 - (\beta\rho)^{D-1}}{1 - \beta\rho}$$

$$+ (c - \bar{l} + \tau)\frac{\beta^2(1-\alpha)(1-\rho)}{1-\beta}\frac{1 - (\beta\rho)^{D-1}}{1 - \beta\rho} \tag{53}$$

$$+ \beta(1-\alpha)\left(B_0 - \frac{c - \bar{l} + \tau}{1 - \beta}\right)(1 - \rho^{D-1}) + (\beta\rho)^{D-1}\beta(1-\alpha)V_u^d.$$

As $V_p^c(B_{D-1}) = V_e^d$ becomes $V_p^c(0) = V_p^d - B_{D-1}$ or using (51):

$$V_p^c(0) = V_e^d - \beta^{-(D-1)}B_0 - \frac{1 - \beta^{-(D-1)}}{1 - \beta}(c - \bar{l} + \tau). \tag{54}$$

Substituting (54) into (53), we obtain:

$$\Psi_1\tau = \Psi_2 - \Psi_3 B_0, \tag{55}$$

with

$$\Psi_1 = \beta^2(1-\alpha)(1-\rho)\frac{1-(\beta\rho)^{D-1}}{1-\beta\rho}\frac{\beta^{-(D-1)}}{1-\beta}$$
$$+ \frac{1}{1-\beta}\left(\rho^{D-1}\beta(1-\alpha) - \beta^{-(D-1)}(1-\beta\alpha)\right),$$

$$\Psi_2 = -\left(u(c)-\bar{l}\right)\left(1+\beta(1-\alpha)\frac{1-(\beta\rho)^{D-1}}{1-\beta\rho}\right)$$
$$- (c-\bar{l})\left(\beta^2(1-\alpha)(1-\rho)\frac{1-(\beta\rho)^{D-1}}{1-\beta\rho}\frac{\beta^{-(D-1)}}{1-\beta}\right.$$
$$\left. + \frac{1}{1-\beta}\left(\rho^{D-1}\beta(1-\alpha) - \beta^{-(D-1)}(1-\beta\alpha)\right)\right)$$
$$\left(1-\beta\alpha-\beta^2(1-\alpha)(1-\rho)\frac{1-(\beta\rho)^{D-1}}{1-\beta\rho}\right)V_e^d - (\beta\rho)^{D-1}\beta(1-\alpha)V_u^d,$$

$$\Psi_3 = \left(1-\beta\alpha-\beta(1-\alpha)\frac{\beta(1-\rho)+(1-\beta)(\beta\rho)^{D-1}}{1-\beta\rho}\right)\beta^{-(D-1)}.$$

Second equation

$V_p^c(0)$ is the welfare of a participating country contributing to the fund. Since V_{-10} is the welfare of a country joining the fund and thus receiving a transfer τ_{-10}. We thus have $V_{-10}(0) = V_p^c(0) + \tau + \tau_{-10}$, as countries joining the fund after default have 0 wealth and receive τ_{-10} instead of paying τ. As we have (by assumption):

$$V_{-10}(0) = \tilde{V}_{-10}(0),$$

we obtain using (54):

$$\tilde{V}_{-10}(0) - \tau_{-10} = V_e^d - \beta^{-(D-1)}B_0 - \frac{1-\beta^{-(D-1)}}{1-\beta}(c-\bar{l}) + \left(1 - \frac{1-\beta^{-(D-1)}}{1-\beta}\right)\tau,$$

which becomes with (55)

$$\tau_{-10} = -V_e^d + \tilde{V}_{-10}(0) + \frac{1-\beta^{-(D-1)}}{1-\beta}(c-\bar{l}) + \left(1 - \frac{1-\beta^{-(D-1)}}{1-\beta}\right)\tau + \beta B_0. \qquad (56)$$

Third equation

Using the expression (51) of B_k in the financial market equilibrium (46), using the expression (36) of n_k, we obtain

$$
\begin{aligned}
\frac{F}{(1-\alpha)n_0} &= \frac{B_0}{1-\alpha} + \sum_{k=1}^{D-1} \rho^{k-1}\left(\frac{B_0}{\beta^k} + (c-\bar{l}+\tau)\frac{1-\beta^{-k}}{1-\beta}\right) \\
&= \frac{B_0}{1-\alpha} + \left(B_0 - \frac{c-\bar{l}+\tau}{1-\beta}\right)\frac{1-\rho^{D-1}\beta^{-(D-1)}}{\beta-\rho} + (c-\bar{l}+\tau)\frac{1-\rho^{D-1}}{(1-\beta)(1-\rho)} \\
&= B_0\left(\frac{1}{1-\alpha} + \frac{1-\rho^{D-1}\beta^{-(D-1)}}{\beta-\rho}\right) + \frac{c-\bar{l}+\tau}{1-\beta}\left(\frac{1-\rho^{D-1}}{1-\rho} - \frac{1-\rho^{D-1}\beta^{-(D-1)}}{\beta-\rho}\right).
\end{aligned}
$$

(57)

Fourth equation

The fourth equation is the budget constraint of the fund (49):

$$
(1-\beta)F = -n_{-10}\tau_{-10} + (n_p - n_{-10})\tau.
$$

(58)

We thus have four linear equations (55), (56), (57), and (58) with four unknowns in B_0, F, τ, τ_{-10}. Except particular cases (which would correspond to a non-invertible matrix for the linear system – which is a zero-measure set), there is always a solution. These four equations insure that there exist a fund economy equilibrium where countries default after $D = \tilde{D}$ periods, where $q = \beta$ and where the fund budget is balanced.

We now prove that the welfare in the fund economy is the same as in the constrained-optimal equilibrium, which is the next Lemma.

Proposition 6 (existence) *If $q = \beta$ and $D = \tilde{D}$, then the welfare in the fund-economy is the same as the welfare in the constrained efficient allocation.*

As before, we denote with tilde the constraint efficient allocation, and without tilde the fund-economy allocation. The welfare in the constrained efficient allocation is simply, with $c = u'^{-1}(1)$ and $c^d = u'^{-1}(1/\varphi)$

$$
\tilde{U} = \sum_{k=-1}^{\tilde{D}-1} \tilde{n}_{k0}\left(u(c) - \tilde{l}_{k0}\right) + \sum_{k=1}^{\tilde{D}-1} \tilde{n}_k\left(u(c) - \bar{l}\right) + \tilde{n}_p^d\left(u\left(c^d\right) - \frac{c^d}{\varphi}\right) + \tilde{n}_u^d\left(u\left(\varphi\bar{l}\right) - \bar{l}\right).
$$

Thus

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$$\tilde{U} = \tilde{n}_p u\left(c\right) - \sum_{k=-1}^{\tilde{D}-1} \tilde{n}_{k0}\tilde{l}_{k0} - \sum_{k=1}^{\tilde{D}-1} \tilde{n}_k \bar{l} + \tilde{n}_p^d\left(u\left(c^d\right) - \frac{c^d}{\varphi}\right) + \tilde{n}_u^d\left(u\left(\varphi\bar{l}\right) - \bar{l}\right). \qquad (59)$$

The goods market equilibrium is

$$\tilde{n}_p c = \sum_{k=-1}^{\tilde{D}-1} \tilde{n}_{k0}\tilde{l}_{k0}^p + \left(\sum_{k=1}^{\tilde{D}-1} \tilde{n}_k\right)\bar{l}. \qquad (60)$$

In the fund-economy, as $q = \beta$, the consumption of all participating countries is c (whereas non-participating countries live in autarky). The welfare is

$$U = n_p u\left(c\right) - \sum_{k=-1}^{D-1} n_{k0}l_{k0} - \sum_{k=1}^{D-1} n_k \bar{l} + n_p^d\left(u\left(c^d\right) - \frac{c^d}{\varphi}\right) + n_u^d\left(u\left(\varphi\bar{l}\right) - \bar{l}\right). \qquad (61)$$

The goods market equilibrium is

$$n_p c = \sum_{k=-1}^{D-1} n_{k0}l_{k0} + \left(\sum_{k=1}^{D-1} n_k\right)\bar{l}. \qquad (62)$$

As $D = \tilde{D}$, we have $n_p = \tilde{n}_p$, $n_{k0} = \tilde{n}_{k0}$, $n_k = \tilde{n}_k$, $n_p^d = \tilde{n}_p^d$ and $n_u^d = \tilde{n}_u^d$. In consequence, (60) and (62) imply that $\sum_{k=-1}^{D-1} n_{k0}l_{k0} = \sum_{k=-1}^{\tilde{D}-1} \tilde{n}_{k0}\tilde{l}_{k0}^p$. We deduce from (59) and (61) that $U = \tilde{U}$.