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# Variational representations for $N$ -cyclically monotone vector fields

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## Abstract

Given a convex bounded domain  $\Omega$  in  $\mathbb{R}^d$  and an integer  $N \geq 2$ , we associate to any *jointly  $N$ -monotone*  $(N - 1)$ -tuple  $(u_1, u_2, \dots, u_{N-1})$  of vector fields from  $\Omega$  into  $\mathbb{R}^d$ , a Hamiltonian  $H$  on  $\mathbb{R}^d \times \mathbb{R}^d \dots \times \mathbb{R}^d$ , that is concave in the first variable, jointly convex in the last  $(N - 1)$  variables such that for almost all  $x \in \Omega$ ,

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x).$$

Moreover,  $H$  is  $N$ -sub-antisymmetric, meaning that  $\sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \leq 0$  for all  $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$ ,  $\sigma$

being the cyclic permutation on  $\mathbb{R}^d$  defined by  $\sigma(x_1, x_2, \dots, x_N) = (x_2, x_3, \dots, x_N, x_1)$ . Furthermore,  $H$  is  $N$ -antisymmetric in a sense to be defined below. This can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator, a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere  $N$ -monotone, showing that they are dual to the class of measure preserving  $N$ -involutions on  $\Omega$ .

## 1 Introduction

Given a domain  $\Omega$  in  $\mathbb{R}^d$ , recall that a single-valued map  $u$  from  $\Omega$  to  $\mathbb{R}^d$  is said to be  *$N$ -cyclically monotone* if for every cycle  $x_1, \dots, x_N, x_{N+1} = x_1$  of points in  $\Omega$ , one has

$$\sum_{i=1}^N \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \quad (1)$$

A classical theorem of Rockafellar [10] states that a map  $u$  from  $\Omega$  to  $\mathbb{R}^d$  is  *$N$ -cyclically monotone for every  $N \geq 2$*  if and only if

$$u(x) \in \partial\phi(x) \text{ for all } x \in \Omega, \quad (2)$$

where  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  is a convex function. On the other hand, a result of E. Krauss [9] yields that  $u$  is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$u(x) \in \partial_2 H(x, x) \text{ for all } x \in \Omega, \quad (3)$$

where  $H$  is a concave-convex antisymmetric Hamiltonian on  $\mathbb{R}^d \times \mathbb{R}^d$ , and  $\partial_2 H$  is the subdifferential of  $H$  as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of  $N$ -cyclically monotone vector fields, where  $N \geq 3$ . We shall give a representation for a family of  $(N - 1)$  vector fields, which may or may not be individually  $N$ -cyclically monotone. Here is the needed concept.

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**Definition 1** Let  $u_1, \dots, u_{N-1}$  be bounded vector fields from a domain  $\Omega \subset \mathbb{R}^d$  into  $\mathbb{R}^d$ . We shall say that the  $(N-1)$ -tuple  $(u_1, u_2, \dots, u_{N-1})$  is jointly  $N$ -monotone, if for every cycle  $x_1, \dots, x_{N+\ell}$  of points in  $\Omega$  such that  $x_{N+i} = x_i$  for  $1 \leq i \leq \ell$ , one has

$$\sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \quad (4)$$

**Examples of jointly  $N$ -monotone families of vector fields:**

- It is clear that  $(u, 0, 0, \dots, 0)$  is jointly  $N$ -monotone if and only if  $u$  is  $N$ -monotone.
- More generally, if each  $u_\ell$  is  $N$ -monotone, then the family  $(u_1, u_2, \dots, u_{N-1})$  is jointly  $N$ -monotone. Actually, one only needs that for  $1 \leq \ell \leq N-1$ , the vector field  $u_\ell$  be  $(N, \ell)$ -monotone, in the following sense: for every cycle  $x_1, \dots, x_{N+\ell}$  of points in  $\Omega$  such that  $x_{N+i} = x_i$  for  $1 \leq i \leq \ell$ , we have

$$\sum_{i=1}^N \langle u_\ell(x_i), x_i - x_{\ell+i} \rangle \geq 0. \quad (5)$$

This notion is sometimes weaker than  $N$ -monotonicity since if  $\ell$  divides  $N$ , then it suffices for  $u$  to be  $\frac{N}{\ell}$ -monotone in order to be an  $(N, \ell)$ -monotone vector field. For example, if  $u_1$  and  $u_3$  are 4-monotone operators and  $u_2$  is 2-monotone, then the triplet  $(u_1, u_2, u_3)$  is jointly 4-monotone.

- Another example is when  $(u_1, u_2, u_3)$  are vector fields such that  $u_2$  is 2-monotone and

$$\langle u_1(x) - u_3(y), x - y \rangle \geq 0 \text{ for every } x, y \in \mathbb{R}^d.$$

In this case, the triplet  $(u_1, u_2, u_3)$  is jointly 4-monotone. In particular, if  $u_1$  and  $u_2$  are both 2-monotone, then the triplet  $(u_1, u_2, u_1)$  is jointly 4-monotone.

- More generally, it is easy to show that  $(u, u, \dots, u)$  is jointly  $N$ -monotone if and only if  $u$  is 2-cyclically monotone.

In the sequel, we shall denote by  $\sigma$  the cyclic permutation on  $\mathbb{R}^d \times \dots \times \mathbb{R}^d$ , defined by

$$\sigma(x_1, x_2, \dots, x_{N-1}, x_N) = (x_2, x_3, \dots, x_N, x_1),$$

and consider the family of continuous  $N$ -antisymmetric Hamiltonians on  $\Omega^N$ , that is

$$\mathcal{H}_N(\Omega) = \{H \in C(\Omega^N); \sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) = 0\} \quad (6)$$

We shall say that  $H$  is  $N$ -sub-antisymmetric on  $\Omega$  if

$$\sum_{i=0}^{N-1} H(\sigma^i(x_1, \dots, x_N)) \leq 0 \text{ on } \Omega^N \quad \text{and} \quad H(x, x, \dots, x) = 0 \text{ on the diagonal.} \quad (7)$$

We shall also say that a function  $F$  of two variables is  $N$ -cyclically sub-antisymmetric on  $\Omega$ , if

$$F(x, x) = 0 \text{ and } \sum_{i=1}^N F(x_i, x_{i+1}) \leq 0 \text{ for all cyclic families } x_1, \dots, x_N, x_{N+1} = x_1 \text{ in } \Omega. \quad (8)$$

Note that if a function  $H(x_1, \dots, x_N)$  is  $N$ -sub-antisymmetric and if it only depends on the first two variables, then the function  $F(x_1, x_2) := H(x_1, x_2, \dots, x_N)$  is  $N$ -cyclically sub-antisymmetric.

We associate to any function  $H$  on  $\Omega^N$ , the following functional on  $\Omega \times (\mathbb{R}^d)^{N-1}$ ,

$$L_H(x, p_1, \dots, p_{N-1}) = \sup \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - H(x, y_1, \dots, y_{N-1}); y_i \in \Omega \right\}. \quad (9)$$

Note that if  $\Omega$  is convex and if  $H$  is convex in the last  $(N-1)$  variables, then  $L_H$  is nothing but the Legendre transform of  $\tilde{H}$  with respect to the last  $(N-1)$  variables, where  $\tilde{H}$  is the extension of  $H$  over  $(\mathbb{R}^d)^N$ , defined as:  $\tilde{H} = H$  on  $\Omega^N$  and  $\tilde{H} = +\infty$  outside of  $\Omega^N$ . Since  $H(x, \dots, x) = 0$  for any  $H \in \mathcal{H}_N(\Omega)$ , then for any such  $H$ , we have for  $x \in \Omega$  and  $p_1, \dots, p_{N-1} \in \mathbb{R}^d$ ,

$$L_H(x, p_1, \dots, p_{N-1}) \geq \sum_{i=1}^{N-1} \langle x, p_i \rangle. \quad (10)$$

To formulate variational principles for such vector fields, we shall consider the class of  $\sigma$ -invariant probability measures on  $\Omega^N$ , which are those  $\pi \in \mathcal{P}(\Omega^N)$  such that for all  $h \in L^1(\Omega^N, d\pi)$ , we have

$$\int_{\Omega^N} h(x_1, \dots, x_N) d\pi = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \quad (11)$$

We denote

$$\mathcal{P}_{\text{sym}}(\Omega^N) = \{\pi \in \mathcal{P}(\Omega^N); \pi \text{ } \sigma\text{-invariant probability on } \Omega^N\}. \quad (12)$$

For a given probability measure  $\mu$  on  $\Omega$ , we also consider the class

$$\mathcal{P}_{\text{sym}}^\mu(\Omega^N) = \{\pi \in \mathcal{P}_{\text{sym}}(\Omega^N); \text{proj}_1 \pi = \mu\}, \quad (13)$$

i.e., the set of all  $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N)$  with a given first marginal  $\mu$ , meaning that

$$\int_{\Omega^N} f(x_1) d\pi(x_1, \dots, x_N) = \int_{\Omega} f(x_1) d\mu(x_1) \text{ for every } f \in L^1(\Omega, \mu). \quad (14)$$

Consider now the set  $\mathcal{S}(\Omega, \mu)$  of  $\mu$ -measure preserving transformations on  $\Omega$ , which can be identified with a closed subset of the sphere of  $L^2(\Omega, \mathbb{R}^d)$ . We shall also consider the subset of  $\mathcal{S}(\Omega, \mu)$  consisting of  $N$ -involutions, that is

$$\mathcal{S}_N(\Omega, \mu) = \{S \in \mathcal{S}(\Omega, \mu); S^N = I \text{ } \mu \text{ a.e.}\}.$$

## 2 Monotone vector fields and $N$ -antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.

**Theorem 2** *Let  $N \geq 2$  be an integer, and consider  $u_1, \dots, u_{N-1}$  to be bounded vector fields from a convex domain  $\Omega \subset \mathbb{R}^d$  into  $\mathbb{R}^d$ .*

1. *If the  $(N-1)$ -tuple  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone, then there exists an  $N$ -sub-antisymmetric Hamiltonian  $H$  that is concave in the first variable, convex in the other  $(N-1)$  variables such that*

$$(u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega. \quad (15)$$

*Moreover,  $H$  is  $N$ -antisymmetric in the following sense*

$$H(x_1, x_2, \dots, x_N) + H_{2, \dots, N}(x_1, x_2, \dots, x_N) = 0, \quad (16)$$

*where  $H_{2, \dots, N}$  is the concavification of the function  $K(\mathbf{x}) = \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))$  with respect to the last  $(N-1)$  variables.*

*Furthermore, there exists a continuous  $N$ -antisymmetric Hamiltonian  $\bar{H}$  on  $\Omega^N$ , such that*

$$L_{\bar{H}}(x, u_1(x), u_2(x), \dots, u_{N-1}(x)) = \sum_{i=1}^{N-1} \langle u_i(x), x \rangle \text{ for all } x \in \Omega. \quad (17)$$

2. *Conversely, if  $(u_1, \dots, u_{N-1})$  satisfy (15) for some  $N$ -sub-antisymmetric Hamiltonian  $H$  that is concave in the first variable, convex in the other variables, then the  $(N-1)$ -tuple  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone.*

**Remark 3** Note that in the case  $N = 2$ ,  $K(\mathbf{x}) = H(x_2, x_1)$  is concave with respect to  $x_2$ , hence  $H_2(x_1, x_2) = H(x_2, x_1)$ , and (16) becomes

$$H(x_1, x_2) + H(x_2, x_1) = 0,$$

thus  $H$  is antisymmetric, recovering well-known results [9], [4], [7], [8].

We start with the following lemma.

**Lemma 4** Assume the  $(N - 1)$ -tuple of bounded vector fields  $(u_1, \dots, u_{N-1})$  on  $\Omega$  is jointly  $N$ -monotone. Let  $f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$  and consider the function  $\tilde{f}(x_1, \dots, x_N)$  to be the convexification of  $f$  with respect to the first variable, that is

$$\tilde{f}(x_1, x_2, \dots, x_N) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) : n \in \mathbb{N}, \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_1^k = x_1 \right\}. \quad (18)$$

Then,  $\tilde{f}$  satisfies the following properties:

1.  $f \geq \tilde{f}$  on  $\Omega^N$ ;
2.  $\tilde{f}$  is convex in the first variable and concave with respect to the other variables;
3.  $\tilde{f}(x, x, \dots, x) = 0$  for each  $x \in \Omega$ ,
4.  $\tilde{f}$  satisfies

$$\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, \dots, x_N)) \geq 0 \text{ on } \Omega^N. \quad (19)$$

**Proof:** Since the  $(N - 1)$ -tuple  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone, it is easy to see that the function

$$f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$$

is linear in the last  $(N - 1)$  variables, that  $f(x, x, \dots, x) = 0$ , and that

$$\sum_{i=0}^{N-1} f(\sigma^i(x_1, \dots, x_N)) \geq 0 \text{ on } \Omega^N. \quad (20)$$

It is also clear that  $f \geq \tilde{f}$ , that  $\tilde{f}$  is convex with respect to the first variable  $x_1$ , and that it is concave with respect to the other variables  $x_2, \dots, x_N$ , since  $f$  itself is concave (actually linear) with respect to  $x_2, \dots, x_N$ . We now show that  $\tilde{f}$  satisfies (19).

For that, we fix  $x_1, x_2, \dots, x_N$  in  $\Omega$  and consider  $(x_1^k)_{k=1}^n$  in  $\Omega$ , and  $(\lambda_k)_k$  in  $\mathbb{R}$  such that  $\lambda_k \geq 0$  such that  $\sum_{k=1}^n \lambda_k = 1$  and  $\sum_{k=1}^n \lambda_k x_1^k = x_1$ . For each  $k$ , we have

$$f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1^k) + \dots + f(x_N, x_1^k, x_2, \dots, x_{N-1}) \geq 0.$$

Multiplying by  $\lambda_k$ , summing over  $k$ , and using that  $f$  is linear in the last  $(N - 1)$ -variables, we have

$$\sum_{k=1}^n \lambda_k f(x_1^k, x_2, \dots, x_N) + f(x_2, \dots, x_N, x_1) + \dots + f(x_N, x_1, x_2, \dots, x_{N-1}) \geq 0.$$

By taking the infimum, we obtain

$$\tilde{f}(x_1, x_2, \dots, x_N) + \sum_{i=1}^{N-1} f(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0.$$

Let now  $n \in \mathbb{N}$ ,  $\lambda_k \geq 0$ ,  $x_N^k \in \Omega$  be such that  $\sum_{k=1}^n \lambda_k = 1$  and  $\sum_{k=1}^n \lambda_k x_2^k = x_2$ . We have for every  $1 \leq k \leq n$ ,

$$\tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0.$$

Multiplying by  $\lambda_k$ , summing over  $k$  and using that  $\tilde{f}$  is convex in the first variable and  $f$  is linear in the last  $(N - 1)$ -variables, we obtain

$$\begin{aligned} & \tilde{f}(x_1, x_2, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \\ \geq & \sum_{k=1}^n \lambda_k \tilde{f}(x_1, x_2^k, x_3, \dots, x_N) + \sum_{k=1}^n \lambda_k f(x_2^k, x_3, \dots, x_1) + \dots + \sum_{k=1}^n \lambda_k f(x_N, x_1, x_2^k, x_3, \dots, x_{N-1}) \geq 0. \end{aligned}$$

By taking the infimum over all possible such choices, we get

$$\tilde{f}(x_1, x_2, x_3, \dots, x_N) + \tilde{f}(x_2, x_3, \dots, x_1) + \dots + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

By repeating this procedure with  $x_3, \dots, x_{N-1}$ , we get

$$\sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) + f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq 0.$$

Finally, since

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)).$$

and since  $\tilde{f}$  is concave in the last  $(N - 1)$  variables, we have for fixed  $x_1, x_2, \dots, x_{N-1}$ , that the function

$$x_N \rightarrow - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N))$$

is a convex minorant of  $x_N \rightarrow f(x_N, x_1, x_2, x_3, \dots, x_{N-1})$ . It follows that

$$f(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq \tilde{f}(x_N, x_1, x_2, x_3, \dots, x_{N-1}) \geq - \sum_{i=0}^{N-2} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)),$$

which finally implies that  $\sum_{i=0}^{N-1} \tilde{f}(\sigma^i(x_1, x_2, \dots, x_N)) \geq 0$ .

This clearly implies that  $\tilde{f}(x, x, \dots, x) \geq 0$  for any  $x \in \Omega$ . On the other hand, since  $\tilde{f}(x, x, \dots, x) \leq f(x, x, \dots, x) = 0$ , we get that  $\tilde{f}(x, x, \dots, x) = 0$  for all  $x \in \Omega$ .  $\square$

**Proof of Theorem 2:** Assume the  $(N - 1)$ -tuple of vector fields  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone on  $\Omega$ , and consider the function  $f(x_1, \dots, x_N) := \sum_{l=1}^{N-1} \langle u_l(x_1), x_1 - x_{l+1} \rangle$  as well as its convexification with respect to the first variable  $\tilde{f}(x_1, \dots, x_N)$ .

By Lemma 4, the function  $\psi(x_1, \dots, x_N) := -\tilde{f}(x_1, \dots, x_N)$  satisfies the following properties

- (i)  $x_1 \rightarrow \psi(x_1, \dots, x_N)$  is concave;
- (ii)  $(x_2, x_3, \dots, x_N) \rightarrow \psi(x_1, \dots, x_N)$  is convex;
- (iii)  $\psi(x_1, \dots, x_N) \geq -f(x_1, \dots, x_N) = \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$ ;
- (iv)  $\psi$  is  $N$ -sub-antisymmetric.

Consider now the family  $\mathcal{H}$  of functions  $H : \Omega^N \rightarrow \mathbb{R}$  such that

1.  $H(x_1, x_2, \dots, x_N) \geq \sum_{l=1}^{N-1} \langle u_l(x_1), x_{l+1} - x_1 \rangle$  for every  $N$ -tuple  $(x_1, \dots, x_N)$  in  $\Omega^N$ ;
2.  $H$  is concave in the first variable;
3.  $H$  is jointly convex in the last  $(N - 1)$  variables;
4.  $H$  is  $N$ -sub-antisymmetric.

Note that  $\mathcal{H} \neq \emptyset$  since  $\psi$  belongs to  $\mathcal{H}$ . Moreover, by  $N$ -subsymmetry, any  $H \in \mathcal{H}$  satisfies for all  $\mathbf{x} = (x_1, \dots, x_N) \in \Omega^N$ ,

$$H(\mathbf{x}) \leq - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})) \leq - \sum_{i=1}^{N-1} \psi(\sigma^i(\mathbf{x})). \quad (21)$$

This also yields that

$$\sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle \leq H(\mathbf{x}) \leq - \sum_{i=2}^N \sum_{\ell=1}^{N-1} \langle u_\ell(x_i), x_i - x_{i+\ell} \rangle, \quad (22)$$

where we denote  $x_{i+N} := x_i$  for  $i = 1, \dots, \ell$ . This yields that  $H(x, x, \dots, x) = 0$  for every  $H \in \mathcal{H}$  and any  $x \in \Omega$ .

On the other hand, it is easy to see that every directed family  $(H_i)_i$  in  $\mathcal{H}$  has a supremum  $H_\infty \in \mathcal{H}$ , meaning that  $\mathcal{H}$  is a Zorn family, and therefore it has a maximal element  $H$ .

Consider now the function

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N},$$

and note that

(i)  $\bar{H}$  is  $N$ -antisymmetric, since

$$\bar{H}(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N-1} [H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))],$$

and each  $K_i(\mathbf{x}) := H(\mathbf{x}) - H(\sigma^i(\mathbf{x}))$  is  $N$ -antisymmetric.

(ii)  $\bar{H} \geq H$  on  $\Omega^N$ , since

$$N[\bar{H}(\mathbf{x}) - H(\mathbf{x})] = - \sum_{i=0}^{N-1} H(\sigma^i(\mathbf{x})) \geq 0,$$

because  $H$  itself is  $N$ -sub-antisymmetric.

The maximality of  $H$  would have implied that  $H = \bar{H}$  is  $N$ -antisymmetric if only  $\bar{H}$  was jointly convex in the last  $(N-1)$ -variables, but since this is not necessarily the case, we consider for  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ , the function

$$K(x_1, x_2, \dots, x_N) = K(\mathbf{x}) := - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x})),$$

which is already concave in the first variable  $x_1$ . Its convexification in the last  $(N-1)$ -variables, that is

$$K^{2, \dots, N}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^n \lambda_i K(x_1, x_2^i, \dots, x_N^i); \lambda_i \geq 0, \sum_{i=1}^n \lambda_i (x_2^i, \dots, x_N^i, 1) = (x_2, \dots, x_N, 1) \right\},$$

is still concave in the first variable, but is now convex in the last  $(N-1)$  variables. Moreover,

$$H \leq K^{2, \dots, N} \leq K = - \sum_{i=1}^{N-1} H \circ \sigma^i. \quad (23)$$

Indeed,  $K^{2, \dots, N} \leq K$  from the definition of  $K^{2, \dots, N}$ , while  $H \leq K^{2, \dots, N}$  because  $H \leq K$  and  $H$  is already convex in the last  $(N-1)$ -variables. It follows that

$$H \leq \frac{(N-1)H + K^{2, \dots, N}}{N} \leq \frac{(N-1)H + K}{N} = \frac{(N-1)H - \sum_{i=1}^{N-1} H \circ \sigma^i}{N} = \bar{H}.$$

The function  $H' = \frac{(N-1)H + K^{2 \cdots N}}{N}$  belongs to the family  $\mathcal{H}$  and therefore  $H = H'$  by the maximality of  $H$ .

This finally yields that  $\bar{H}$  is  $N$ -sub-antisymmetric, that  $H(x, x, x) = 0$  for all  $x \in \Omega$  and that

$$H(\mathbf{x}) + H_{2, \dots, N}(\mathbf{x}) = 0 \text{ for every } \mathbf{x} \in \Omega^N,$$

where  $H_{2, \dots, N} = -K^{2 \cdots N}$ , which for a fixed  $x_1$ , is nothing but the concavification of  $(x_2, \dots, x_N) \rightarrow \sum_{i=1}^{N-1} H(\sigma^i(x_1, x_2, \dots, x_N))$ .

Note now that since for any  $x_1, \dots, x_N$  in  $\Omega$ ,

$$H(x_1, x_2, \dots, x_N) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle, \quad (24)$$

and

$$H(x_1, x_1, \dots, x_1) = 0, \quad (25)$$

we have

$$H(x_1, x_2, \dots, x_N) - H(x_1, \dots, x_1) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} - x_1 \rangle. \quad (26)$$

Since  $H$  is convex in the last  $(N-1)$  variables, this means that for all  $x \in \Omega$ , we have

$$(u_1(x), u_2(x), \dots, u_{N-1}(x)) \in \partial_{2, \dots, N} H(x, x, \dots, x). \quad (27)$$

as claimed in (15). Note that this also yield that

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \text{ for all } x \in \Omega.$$

In other words,  $L_H(x, u_1(x), \dots, u_{N-1}(x)) = \sum_{\ell=1}^{N-1} \int_\Omega \langle u_\ell(x), x \rangle$  for all  $x \in \Omega$ . As above, consider

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

We have that  $\bar{H} \in \mathcal{H}_N(\Omega)$  and  $\bar{H} \geq H$ , and therefore  $L_{\bar{H}} \leq L_H$ . On the other hand, we have for all  $x \in \Omega$ ,

$$L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) = L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) + \bar{H}(x, x, \dots, x) \geq \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle.$$

To prove (17), we use the appendix in [6] to deduce that for  $i = 2, \dots, N$ , the gradients  $\nabla_i H(x, x, \dots, x)$  actually exist for a.e.  $x$  in  $\Omega$ .

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone follows from (24) and the sub-antisymmetry of  $H$ .  $\square$

In the case of a single  $N$ -monotone vector field, we can obviously apply the above theorem to the  $(N-1)$ -tuple  $(u, 0, \dots, 0)$  which is then  $N$ -monotone to find a  $N$ -sub-antisymmetric Hamiltonian  $H$ , which is concave in the first variable, convex in the last  $(N-1)$  variables such that

$$(-u(x), u(x), 0, \dots, 0) = \nabla H(x, x, \dots, x) \text{ for a.e. } x \in \Omega. \quad (28)$$

However, in this case we can restrict ourselves to  $N$ -cyclically sub-antisymmetric functions of two variables and establish the following extension of the Theorem of Krauss.

**Theorem 5** *If  $u$  is  $N$ -cyclically monotone on  $\Omega$ , then there exists a concave-convex function of two variables  $F$  that is  $N$ -cyclically sub-antisymmetric, such that*

$$(-u(x), u(x)) \in \partial F(x, x) \text{ for all } x \in \Omega, \quad (29)$$

where  $\partial H$  is the sub-differential of  $H$  as a concave-convex function [11]. Moreover,

$$u(x) = \nabla_2 F(x, x) \text{ for a.e. } x \in \Omega. \quad (30)$$

**Proof:** Let  $f(x, y) = \langle u(x), x - y \rangle$  and let  $f^1(x, y)$  be its convexification in  $x$  for fixed  $y$ , that is

$$f^1(x, y) = \inf \left\{ \sum_{k=1}^n \lambda_k f(x_k, y) : \lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1, \sum_{k=1}^n \lambda_k x_k = x \right\}. \quad (31)$$

Since  $f(x, x) = 0$ ,  $f$  is linear in  $y$ , and  $\sum_{i=1}^N f(x_i, x_{i+1}) \geq 0$  for any cyclic family  $x_1, \dots, x_N, x_{N+1} = x_1$  in  $\Omega$ , it is easy to show that  $f \geq f^1$  on  $\Omega$ ,  $f^1$  is convex in the first variable and concave with respect to the second,  $f^1(x, x) = 0$  for each  $x \in \Omega$ , and that  $f^1$  is  $N$ -cyclically supersymmetric in the sense that for any cyclic family  $x_1, \dots, x_N, x_{N+1} = x_1$  in  $\Omega$ , we have  $\sum_{i=1}^N f^1(x_i, x_{i+1}) \geq 0$ .

Consider now  $F(x, y) = -f^1(x, y)$  and note that  $x \rightarrow F(x, y)$  is concave,  $y \rightarrow F(x, y)$  is convex,  $F(x, y) \geq -f(x, y) = \langle u(x), y - x \rangle$  and  $F$  is  $N$ -cyclically sub-antisymmetric. By the antisymmetry, we have

$$\langle u(x_1), x_2 - x_1 \rangle \leq F(x_1, x_2) \leq \langle u(x_2), x_2 - x_1 \rangle, \quad (32)$$

which yields that  $(-u(x), u(x)) \in \partial F(x, x)$  for all  $x \in \Omega$ .

Since  $F$  is anti-symmetric and concave-convex, the possibly multivalued map  $x \rightarrow \partial_2 F(x, x)$  is monotone on  $\Omega$ , and therefore single-valued and differentiable almost everywhere [10]. This completes the proof.

**Remark 6** Note that we cannot expect to have a function  $F$  such that  $\sum_{i=1}^N F(x_i, x_{i+1}) = 0$  for all cyclic families  $x_1, \dots, x_N, x_{N+1} = x_1$  in  $\Omega$ . Actually, we believe that the only function satisfying such an  $N$ -antisymmetry for  $N \geq 3$  must be of the form  $F(x, y) = f(x) - f(y)$ . This is the reason why one needs to consider functions of  $N$ -variables in order to get  $N$ -antisymmetry. In other words, the function defined by

$$H(x_1, x_2, \dots, x_N) := \frac{(N-1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1})}{N}, \quad (33)$$

is  $N$ -antisymmetric in the sense of (6) and  $H(x_1, x_2, \dots, x_N) \geq F(x_1, x_2)$  for all  $(x_1, x_2, \dots, x_N)$  in  $\Omega^N$ .

### 3 Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that  $d\mu$  is Lebesgue measure  $dx$  normalized to be a probability on  $\Omega$ . We shall also assume that  $\Omega$  is convex and that its boundary has measure zero.

**Theorem 7** Let  $u_1, \dots, u_{N-1} : \Omega \rightarrow \mathbb{R}^d$  be bounded measurable vector fields. The following properties are then equivalent:

1. The  $(N-1)$ -tuple  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone a.e., that is there exists a measure zero set  $\Omega_0$  such that  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone on  $\Omega \setminus \Omega_0$ .
2. The infimum of the following Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, x_2, \dots, x_N); \pi \in \mathcal{P}_{\text{sym}}^\mu(\Omega^N) \right\} \quad (34)$$

is equal to zero, and is therefore attained by the push-forward of  $\mu$  by the map  $x \rightarrow (x, x, \dots, x)$ .

3.  $(u_1, \dots, u_{N-1})$  is in the polar of  $\mathcal{S}_N(\Omega, \mu)$  in the following sense,

$$\inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0. \quad (35)$$

4. The following holds:

$$\inf \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} |u_\ell(x) - S^\ell x|^2 d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = \sum_{\ell=1}^{N-1} \int_{\Omega} |u_\ell(x) - x|^2 d\mu. \quad (36)$$

5. There exists a  $N$ -sub-antisymmetric Hamiltonian  $H$  which is concave in the first variable, convex in the last  $(N - 1)$  variables such that

$$(u_1(x), \dots, u_{N-1}(x)) = \nabla_{2, \dots, N} H(x, x, \dots, x) \quad \text{for a.e. } x \in \Omega. \quad (37)$$

Moreover,  $H$  is  $N$ -symmetric in the sense of (16).

6. The following duality holds:

$$\inf \left\{ \int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu; H \in \mathcal{H}_N(\Omega) \right\} = \sup \left\{ \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell} x \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\}$$

and the latter is attained at the identity map.

We start with the following lemma, which identifies those probabilities in  $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$  that are carried by graphs of functions from  $\Omega$  to  $\Omega^N$ .

**Lemma 8** *Let  $S : \Omega \rightarrow \Omega$  be a  $\mu$ -measurable map, then the following properties are equivalent:*

1. The image of  $\mu$  by the map  $x \rightarrow (x, Sx, \dots, S^{N-1}x)$  belongs to  $\mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$ .
2.  $S$  is  $\mu$ -measure preserving and  $S^N(x) = x$   $\mu$ -a.e.
3. For any bounded Borel measurable  $N$ -antisymmetric  $H$  on  $\Omega^N$ , we have  $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$ .

**Proof.** It is clear that 1) implies 3) since  $\int_{\Omega^N} H(\mathbf{x}) d\pi(\mathbf{x}) = 0$  for any  $N$ -antisymmetric Hamiltonian  $H$  and any  $\pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N)$ .

That 2) implies 1) is also straightforward since if  $\pi$  is the push-forward of  $\mu$  by a map of the form  $x \rightarrow (x, Sx, \dots, S^{N-1}x)$ , where  $S$  is a  $\mu$ -measure preserving  $S$  with  $S^N x = x$   $\mu$  a.e. on  $\Omega$ , then for all  $h \in L^1(\Omega^N, d\pi)$ , we have

$$\begin{aligned} \int_{\Omega^N} h(x_1, \dots, x_N) d\pi &= \int_{\Omega^N} h(x, Sx, \dots, S^{N-1}x) d\mu(x) = \int_{\Omega^N} h(Sx, S^2x, \dots, S^{N-1}x, S^N x) d\mu(x) \\ &= \int_{\Omega^N} h(Sx, S^2x, \dots, S^{N-1}x, x) d\mu(x) = \int_{\Omega^N} h(\sigma(x_1, \dots, x_N)) d\pi. \end{aligned}$$

We now prove that 2) and 3) are equivalent. Assuming first that  $S$  is  $\mu$ -measure preserving such that  $S^N = I$   $\mu$  a.e., then for every Borel bounded  $N$ -antisymmetric  $H$ , we have

$$\begin{aligned} \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu &= \int_{\Omega} H(Sx, S^2x, \dots, S^{N-1}x, x) d\mu \\ &= \dots = \int_{\Omega} H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) d\mu. \end{aligned}$$

Since  $H$  is  $N$ -antisymmetric, we can see that

$$H(x, Sx, \dots, S^{N-1}x) + H(Sx, S^2x, \dots, S^{N-1}x, x) + \dots H(S^{N-1}x, x, Sx, \dots, S^{N-2}x) = 0.$$

It follows that  $N \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$ .

For the reverse implication, assume  $\int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = 0$  for every  $N$ -antisymmetric Hamiltonian  $H$ . By testing this identity with the Hamiltonians

$$H(x_1, x_2, \dots, x_N) = f(x_1) - f(x_i),$$

where  $f$  is any continuous function on  $\Omega$ , one gets that  $S$  is  $\mu$ -measure preserving. Now take the Hamiltonian

$$H(x_1, x_2, \dots, x_N) = |x_1 - Sx_N| - |Sx_1 - x_2| - |x_2 - Sx_1| + |Sx_2 - x_3|.$$

Note that  $H \in \mathcal{H}_N(\Omega)$  since it is of the form  $H(x_1, \dots, x_N) = f(x_1, x_2, x_N) - f(x_2, x_3, x_1)$ . Now test the above identity with such an  $H$  to obtain

$$0 = \int_{\Omega} H(x, Sx, S^2x, \dots, S^{N-1}x) d\mu = \int_{\Omega} |x - SS^{N-1}x| d\mu.$$

It follows that  $S^N = I$   $\mu$  a.e. on  $\omega$ , and we are done.  $\square$

**Proof of Theorem 7:** To show that (1) implies (2), it suffices to notice that if  $\pi$  is a  $\sigma$ -invariant probability measure on  $\Omega^N$  such that  $\text{proj}_1 \pi = \mu$ , then

$$\begin{aligned} \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) &= \frac{1}{N} \sum_{i=1}^N \int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_i), x_i - x_{i+\ell} \rangle d\pi(x_1, \dots, x_N) \\ &= \frac{1}{N} \int_{\Omega^N} \left( \sum_{i=1}^N \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_i), x_i - x_{i+\ell} \rangle \right) d\pi(x_1, \dots, x_N) \\ &\geq 0, \end{aligned}$$

since  $(u_1, \dots, u_{N-1})$  is jointly  $N$ -monotone. On the other hand, if  $\pi$  is the  $\sigma$ -invariant measure obtained by taking the image of  $\mu := dx$  by  $x \rightarrow (x, \dots, x)$ , then

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi(x_1, \dots, x_N) = 0.$$

To show that (2) implies (3), let  $S$  be a  $\mu$ -measure preserving transformation on  $\Omega$  such that  $S^N = I$   $\mu$  a.e. on  $\Omega$ . Then the image  $\pi_S$  of  $\mu$  by the map

$$x \rightarrow (x, Sx, S^2x, \dots, S^{N-1}x)$$

is  $\sigma$ -invariant, hence

$$\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x_1), x_1 - x_{\ell+1} \rangle d\pi_S(x_1, \dots, x_N) = \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S^{\ell}x \rangle d\mu \geq 0.$$

By taking  $S = I$ , we get that the infimum is necessarily zero.

The equivalence of (3) and (4) follows immediately from developing the square.

We now show that (3) implies (1). For that take  $N$  points  $x_1, x_2, \dots, x_N$  in  $\Omega$ , and let  $R > 0$  be such that  $B(x_i, R) \subset \Omega$ . Consider the transformation

$$S_R(x) = \begin{cases} x - x_1 + x_2 & \text{for } x \in B(x_1, R) \\ x - x_2 + x_3 & \text{for } x \in B(x_2, R) \\ \dots & \\ x - x_N + x_1 & \text{for } x \in B(x_N, R) \\ x & \text{otherwise} \end{cases}$$

It is easy to see that  $S_R$  is a measure preserving transformation and that  $S_R^N = Id$ . We then have

$$0 \leq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x - S_R^{\ell}x \rangle d\mu \leq \sum_{i=1}^N \int_{B(x_i, R)} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu.$$

Letting  $R \rightarrow 0$ , we get from Lebesgue's density theorem, that

$$\frac{1}{|B(x_i, R)|} \int_{B(x_i, R)} \langle u_{\ell}(x), x_i - x_{\ell+i} \rangle d\mu \rightarrow \langle u_{\ell}(x_i), x_i - x_{\ell+i} \rangle,$$

from which follows that  $(u_1, \dots, u_{N-1})$  are jointly  $N$ -monotone a.e. on  $\Omega$ .

The fact that (1) is equivalent to (5) follows immediately from Theorem 2. To prove that 5) implies 6) note that for all  $p_i \in \mathbb{R}^d, x \in \Omega, y_i \in \Omega, i = 1, \dots, N-1$ ,

$$L_H(x, p_1, \dots, p_{N-1}) + H(x, y_1, \dots, y_{N-1}) \geq \sum_{i=1}^{N-1} \langle p_i, y_i \rangle,$$

which yields that for any  $S \in \mathcal{S}_N(\Omega, \mu)$ ,

$$\int_{\Omega} [L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu + H(x, Sx, \dots, S^{N-1}x)] d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If  $H \in \mathcal{H}_N(\Omega)$  and  $S \in \mathcal{S}_N(\Omega, \mu)$ , we then have  $\int_{\Omega} H(x, Sx, \dots, S^{N-1}x) d\mu = 0$ , and therefore

$$\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), S^{\ell}x \rangle d\mu.$$

If now  $H$  is the  $N$ -sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable, convex in the last  $(N-1)$  variables, then

$$L_H(x, u_1(x), \dots, u_{N-1}(x)) + H(x, x, \dots, x) = \sum_{\ell=1}^{N-1} \langle u_{\ell}(x), x \rangle \quad \text{for all } x \in \Omega \setminus \Omega_0,$$

and therefore  $\int_{\Omega} L_H(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$ .

Consider now

$$\bar{H}(\mathbf{x}) = \frac{(N-1)H(\mathbf{x}) - \sum_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))}{N}.$$

As before, we have that  $\bar{H} \in \mathcal{H}_N(\Omega)$  and  $\bar{H} \geq H$ . Since  $L_{\bar{H}} \leq L_H$ , we have that  $\int_{\Omega} L_{\bar{H}}(x, u_1(x), \dots, u_{N-1}(x)) d\mu = \sum_{\ell=1}^{N-1} \int_{\Omega} \langle u_{\ell}(x), x \rangle d\mu$  and (6) is proved.

Finally, note that (6) readily implies (3), which means that  $(u_1, \dots, u_{N-1})$  is then jointly  $N$ -monotone.  $\square$

We now consider again the case of a single  $N$ -cyclically monotone vector field.

**Corollary 9** *Let  $u : \Omega \rightarrow \mathbb{R}^d$  be a bounded measurable vector field. The following properties are then equivalent:*

1.  $u$  is  $N$ -cyclically monotone a.e., that is there exists a measure zero set  $\Omega_0$  such that  $u$  is  $N$ -cyclically monotone on  $\Omega \setminus \Omega_0$ .
2. The infimum of the following Monge-Kantorovich problem

$$\inf \left\{ \int_{\Omega^N} \langle u(x_1), x_1 - x_2 \rangle d\pi(\mathbf{x}); \pi \in \mathcal{P}_{\text{sym}}^{\mu}(\Omega^N) \right\} \quad (38)$$

is equal to zero, and is therefore attained by the push-forward of  $\mu$  by the map  $x \rightarrow (x, x, \dots, x)$ .

3. The vector field  $u$  is in the polar of  $\mathcal{S}_N(\Omega, \mu)$ , that is

$$\inf \left\{ \int_{\Omega} \langle u(x), x - Sx \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0. \quad (39)$$

4. The projection of  $u$  on  $\mathcal{S}_N(\Omega, \mu)$  is the identity map, that is

$$\inf \left\{ \int_{\Omega} |u(x) - Sx|^2 d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = \int_{\Omega} |u(x) - x|^2 d\mu. \quad (40)$$

5. There exists a  $N$ -cyclically sub-antisymmetric function  $H$  of two variables, which is concave in the first variable, convex in the second variable such that

$$u(x) = \nabla_2 H(x, x) \quad \text{for a.e. } x \in \Omega. \quad (41)$$

6. The following duality holds:

$$\inf \left\{ \int_{\Omega} L_H(x, u(x), 0, \dots, 0) d\mu; H \in \mathcal{H}_N(\Omega) \right\} = \sup \left\{ \int_{\Omega} \langle u(x), Sx \rangle d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\}$$

and the latter is attained at the identity map.

**Proof:** This is an immediate application of Theorem 7 applied to the  $(N-1)$ -tuple vector fields  $(u, 0, \dots, 0)$ , which is clearly jointly  $N$ -monotone on  $\Omega \setminus \Omega_0$ , whenever  $u$  is  $N$ -monotone on  $\Omega \setminus \Omega_0$ .

**Remark 10** Note that the sets of  $\mu$ -measure preserving  $N$ -involutions  $(\mathcal{S}_N(\Omega, \mu))_N$  do not form a nested family, that is  $\mathcal{S}_N(\Omega, \mu)$  is not necessarily included in  $\mathcal{S}_M(\Omega, \mu)$ , whenever  $N \leq M$ , unless of course  $M$  is a multiple of  $N$ . On the other hand, the above theorem shows that their polar sets, i.e.,

$$\mathcal{S}_N(\Omega, \mu)^0 = \{u \in L^2(\Omega, \mathbb{R}^d); \int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0 \text{ for all } S \in \mathcal{S}_N(\Omega, \mu)\},$$

which coincide with the  $N$ -cyclically monotone maps, satisfy

$$\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0,$$

for every  $N \geq 1$ . This can also be seen directly. Indeed, it is clear that a 2-involution is a 4-involution but not necessarily a 3-involution. On the other hand, assume that  $u$  is 3-cyclically monotone operator, then for any transformation  $S : \Omega \rightarrow \Omega$ , we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(Sx), Sx - S^2x \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0.$$

If now  $S$  is measure preserving, we have

$$\int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(x), x - Sx \rangle d\mu + \int_{\Omega} \langle u(S^2x), S^2x - x \rangle d\mu \geq 0,$$

and if  $S^2 = I$ , then  $\int_{\Omega} \langle u(x), x - Sx \rangle d\mu \geq 0$ , which means that  $u \in \mathcal{S}_2(\Omega, \mu)^0$ . Similarly, one can show that any  $(N+1)$ -cyclically monotone operator belongs to  $\mathcal{S}_N(\Omega, \mu)^0$ . In other words,  $\mathcal{S}_{N+1}(\Omega, \mu)^0 \subset \mathcal{S}_N(\Omega, \mu)^0$  for all  $N \geq 2$ . Note that  $\mathcal{S}_1(\Omega, \mu)^0 = \{I\}^0 = L^2(\Omega, \mathbb{R}^d)$ , while

$$\mathcal{S}(\Omega, \mu)^0 = \cap_N \mathcal{S}_N(\Omega, \mu)^0 = \{u \in L^2(\Omega, \mathbb{R}^d), u = \nabla \phi \text{ for some convex function } \phi \text{ in } W^{1,2}(\mathbb{R}^d)\},$$

in view of classical results of Rockafellar [11] and Brenier [1].

**Remark 11** In a forthcoming paper [6], the above result is extended to give a similar decomposition for any family of bounded measurable vector fields  $u_1, u_2, \dots, u_{N-1}$  on  $\Omega$ . It is shown there that there exists a measure preserving  $N$ -involution  $S$  on  $\Omega$  and an  $N$ -antisymmetric Hamiltonian  $H$  on  $\Omega^N$  such that for  $i = 1, \dots, N-1$ , we have

$$u_i(x) = \nabla_{i+1} H(x, Sx, S^2x, \dots, S^{N-1}x) \quad \text{for a.e. } x \in \Omega.$$

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