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# Variational representations for $N$-cyclically monotone vector fields 

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#### Abstract

Given a convex bounded domain $\Omega$ in $\mathbb{R}^{d}$ and an integer $N \geq 2$, we associate to any jointly $N$ monotone $(N-1)$-tuplet $\left(u_{1}, u_{2}, \ldots, u_{N-1}\right)$ of vector fields from $\Omega$ into $\mathbb{R}^{d}$, a Hamiltonian $H$ on $\mathbb{R}^{d} \times$ $\mathbb{R}^{d} \ldots \times \mathbb{R}^{d}$, that is concave in the first variable, jointly convex in the last $(N-1)$ variables such that for almost all $x \in \Omega$, $$
\left(u_{1}(x), u_{2}(x), \ldots, u_{N-1}(x)\right)=\nabla_{2, \ldots, N} H(x, x, \ldots, x)
$$ $$
N-1
$$

Moreover, $H$ is $N$-sub-antisymmetric, meaning that $\sum_{i=0}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right) \leq 0$ for all $\mathbf{x}=\left(x_{1}, \ldots, x_{N}\right) \in \Omega^{N}, \sigma$ being the cyclic permutation on $\mathbb{R}^{d}$ defined by $\sigma\left(x_{1}, x_{2}, \ldots, x_{N}\right)=\left(x_{2}, x_{3}, \ldots, x_{N}, x_{1}\right)$. Furthermore, $H$ is $N$-antisymmetric in a sense to be defined below. This can be seen as an extension of a theorem of E. Krauss, which associates to any monotone operator, a concave-convex antisymmetric saddle function. We also give various variational characterizations of vector fields that are almost everywhere $N$-monotone, showing that they are dual to the class of measure preserving $N$-involutions on $\Omega$. .


## 1 Introduction

Given a domain $\Omega$ in $\mathbb{R}^{d}$, recall that a single-valued map $u$ from $\Omega$ to $\mathbb{R}^{d}$ is said to be $N$-cyclically monotone if for every cycle $x_{1}, \ldots, x_{N}, x_{N+1}=x_{1}$ of points in $\Omega$, one has

$$
\begin{equation*}
\sum_{i=1}^{N}\left\langle u\left(x_{i}\right), x_{i}-x_{i+1}\right\rangle \geq 0 \tag{1}
\end{equation*}
$$

A classical theorem of Rockafellar [10] states that a map $u$ from $\Omega$ to $\mathbb{R}^{d}$ is $N$-cyclically monotone for every $N \geq 2$ if and only if

$$
\begin{equation*}
u(x) \in \partial \phi(x) \text { for all } x \in \Omega \tag{2}
\end{equation*}
$$

where $\phi: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is a convex function. On the other hand, a result of E . Krauss 9 yields that $u$ is a monotone map, i.e., a 2-cyclically monotone map, if and only if

$$
\begin{equation*}
u(x) \in \partial_{2} H(x, x) \text { for all } x \in \Omega \tag{3}
\end{equation*}
$$

where $H$ is a concave-convex antisymmetric Hamiltonian on $\mathbb{R}^{d} \times \mathbb{R}^{d}$, and $\partial_{2} H$ is the subdifferential of $H$ as a convex function in the second variable.

In this paper, we extend the result of Krauss to the class of $N$-cyclically monotone vector fields, where $N \geq 3$. We shall give a representation for a family of $(N-1)$ vector fields, which may or may not be individually $N$-cyclically monotone. Here is the needed concept.

[^0]Definition 1 Let $u_{1}, \ldots, u_{N-1}$ be bounded vector fields from a domain $\Omega \subset \mathbb{R}^{d}$ into $\mathbb{R}^{d}$. We shall say that the $(N-1)$-tuple $\left(u_{1}, u_{2}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone, if for every cycle $x_{1}, \ldots, x_{N+\ell}$ of points in $\Omega$ such that $x_{N+i}=x_{i}$ for $1 \leq i \leq \ell$, one has

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{\ell=1}^{N-1}\left\langle u_{l}\left(x_{i}\right), x_{i}-x_{l+i}\right\rangle \geq 0 \tag{4}
\end{equation*}
$$

## Examples of jointly $N$-monotone families of vector fields:

- It is clear that $(u, 0,0, \ldots, 0)$ is jointly $N$-monotone if and only if $u$ is $N$-monotone.
- More generally, if each $u_{\ell}$ is $N$-monotone, then the family $\left(u_{1}, u_{2}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone. Actually, one only needs that for $1 \leq \ell \leq N-1$, the vector field $u_{\ell}$ be $(N, \ell)$-monotone, in the following sense: for every cycle $x_{1}, \ldots, x_{N+\ell}$ of points in $\Omega$ such that $x_{N+i}=x_{i}$ for $1 \leq i \leq \ell$, we have

$$
\begin{equation*}
\sum_{i=1}^{N}\left\langle u_{\ell}\left(x_{i}\right), x_{i}-x_{\ell+i}\right\rangle \geq 0 \tag{5}
\end{equation*}
$$

This notion is sometimes weaker than $N$-monotonicity since if $\ell$ divides $N$, then it suffices for $u$ to be $\frac{N}{\ell}$-monotone in order to be an $(N, \ell)$-monotone vector field. For example, if $u_{1}$ and $u_{3}$ are 4 -monotone operators and $u_{2}$ is 2 -monotone, then the triplet $\left(u_{1}, u_{2}, u_{3}\right)$ is jointly 4 -monotone.

- Another example is when $\left(u_{1}, u_{2}, u_{3}\right)$ are vector fields such that $u_{2}$ is 2 -monotone and

$$
\left\langle u_{1}(x)-u_{3}(y), x-y\right\rangle \geq 0 \text { for every } x, y \in \mathbb{R}^{d} .
$$

In this case, the triplet $\left(u_{1}, u_{2}, u_{3}\right)$ is jointly 4 -monotone. In particular, if $u_{1}$ and $u_{2}$ are both 2 monotone, then the triplet $\left(u_{1}, u_{2}, u_{1}\right)$ is jointly 4-monotone.

- More generally, it is easy to show that $(u, u, \ldots, u)$ is jointly $N$-monotone if and only if $u$ is 2-cyclically monotone.

In the sequel, we shall denote by $\sigma$ the cyclic permutation on $\mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d}$, defined by

$$
\sigma\left(x_{1}, x_{2}, \ldots, x_{N-1}, x_{N}\right)=\left(x_{2}, x_{3}, \ldots, x_{N}, x_{1}\right)
$$

and consider the family of continuous $N$-antisymmetric Hamiltonians on $\Omega^{N}$, that is

$$
\begin{equation*}
\mathcal{H}_{N}(\Omega)=\left\{H \in C\left(\Omega^{N}\right) ; \sum_{i=0}^{N-1} H\left(\sigma^{i}\left(x_{1}, \ldots, x_{N}\right)\right)=0\right\} \tag{6}
\end{equation*}
$$

We shall say that $H$ is $N$-sub-antisymmetric on $\Omega$ if

$$
\begin{equation*}
\sum_{i=0}^{N-1} H\left(\sigma^{i}\left(x_{1}, \ldots, x_{N}\right)\right) \leq 0 \text { on } \Omega^{N} \text { and } H(x, x, \ldots, x)=0 \text { on the diagonal. } \tag{7}
\end{equation*}
$$

We shall also say that a function $F$ of two variables is $N$-cyclically sub-antisymmetric on $\Omega$, if

$$
\begin{equation*}
F(x, x)=0 \text { and } \sum_{i=1}^{N} F\left(x_{i}, x_{i+1}\right) \leq 0 \text { for all cyclic families } x_{1}, \ldots, x_{N}, x_{N+1}=x_{1} \text { in } \Omega \tag{8}
\end{equation*}
$$

Note that if a function $H\left(x_{1}, \ldots, x_{N}\right) N$-sub-antisymmetric and if it only depends on the first two variables, then the function $F\left(x_{1}, x_{2}\right):=H\left(x_{1}, x_{2}, \ldots, x_{N}\right)$ is $N$-cyclically sub-antisymmetric.

We associate to any function $H$ on $\Omega^{N}$, the following functional on $\Omega \times\left(\mathbb{R}^{d}\right)^{N-1}$,

$$
\begin{equation*}
L_{H}\left(x, p_{1}, \ldots, p_{N-1}\right)=\sup \left\{\sum_{i=1}^{N-1}\left\langle p_{i}, y_{i}\right\rangle-H\left(x, y_{1}, \ldots, y_{N-1}\right) ; y_{i} \in \Omega\right\} \tag{9}
\end{equation*}
$$

Note that if $\Omega$ is convex and if $H$ is convex in the last $(N-1)$ variables, then $L_{H}$ is nothing but the Legendre transform of $\tilde{H}$ with respect to the last $(N-1)$ variables, where $\tilde{H}$ is the extension of $H$ over $\left(\mathbb{R}^{d}\right)^{N}$, defined as: $\tilde{H}=H$ on $\Omega^{N}$ and $\tilde{H}=+\infty$ outside of $\Omega^{N}$. Since $H(x, \ldots, x)=0$ for any $H \in \mathcal{H}_{N}(\Omega)$, then for any such $H$, we have for $x \in \Omega$ and $p_{1}, \ldots, p_{N-1} \in \mathbb{R}^{d}$,

$$
\begin{equation*}
L_{H}\left(x, p_{1}, \ldots, p_{N-1}\right) \geq \sum_{i=1}^{N-1}\left\langle x, p_{i}\right\rangle \tag{10}
\end{equation*}
$$

To formulate variational principles for such vector fields, we shall consider the class of $\sigma$-invariant probability measures on $\Omega^{N}$, which are those $\pi \in \mathcal{P}\left(\Omega^{N}\right)$ such that for all $h \in L^{1}\left(\Omega^{N}, d \pi\right)$, we have

$$
\begin{equation*}
\int_{\Omega^{N}} h\left(x_{1}, \ldots, x_{N}\right) d \pi=\int_{\Omega^{N}} h\left(\sigma\left(x_{1}, \ldots, x_{N}\right)\right) d \pi \tag{11}
\end{equation*}
$$

We denote

$$
\begin{equation*}
\mathcal{P}_{\text {sym }}\left(\Omega^{N}\right)=\left\{\pi \in \mathcal{P}\left(\Omega^{N}\right) ; \pi \sigma \text {-invariant probability on } \Omega^{N}\right\} \tag{12}
\end{equation*}
$$

For a given probability measure $\mu$ on $\Omega$, we also consider the class

$$
\begin{equation*}
\mathcal{P}_{\mathrm{sym}}^{\mu}\left(\Omega^{N}\right)=\left\{\pi \in \mathcal{P}_{\mathrm{sym}}\left(\Omega^{N}\right) ; \operatorname{proj}_{1} \pi=\mu\right\}, \tag{13}
\end{equation*}
$$

i.e., the set of all $\pi \in \mathcal{P}_{\text {sym }}\left(\Omega^{N}\right)$ with a given first marginal $\mu$, meaning that

$$
\begin{equation*}
\int_{\Omega^{N}} f\left(x_{1}\right) d \pi\left(x_{1}, \ldots, x_{N}\right)=\int_{\Omega} f\left(x_{1}\right) d \mu\left(x_{1}\right) \text { for every } f \in L^{1}(\Omega, \mu) \tag{14}
\end{equation*}
$$

Consider now the set $\mathcal{S}(\Omega, \mu)$ of $\mu$-measure preserving transformations on $\Omega$, which can be identified with a closed subset of the sphere of $L^{2}\left(\Omega, \mathbb{R}^{d}\right)$. We shall also consider the subset of $\mathcal{S}(\Omega, \mu)$ consisting of $N$ involutions, that is

$$
\mathcal{S}_{N}(\Omega, \mu)=\left\{S \in \mathcal{S}(\Omega, \mu) ; S^{N}=I \mu \text { a.e. }\right\}
$$

## 2 Monotone vector fields and N -antisymmetric Hamiltonians

In this section, we establish the following extension of a theorem of Krauss.
Theorem 2 Let $N \geq 2$ be an integer, and consider $u_{1}, \ldots, u_{N-1}$ to be bounded vector fields from a convex domain $\Omega \subset \mathbb{R}^{d}$ into $\mathbb{R}^{d}$.

1. If the $(N-1)$-tuple $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone, then there exists an $N$-sub-antisymmetric Hamiltonian $H$ that is concave in the first variable, convex in the other $(N-1)$ variables such that

$$
\begin{equation*}
\left(u_{1}(x), \ldots, u_{N-1}(x)\right)=\nabla_{2, \ldots, N} H(x, x, \ldots, x) \quad \text { for a.e. } x \in \Omega . \tag{15}
\end{equation*}
$$

Moreover, $H$ is $N$-antisymmetric in the following sense

$$
\begin{equation*}
H\left(x_{1}, x_{2}, \ldots, x_{N}\right)+H_{2, \ldots, N}\left(x_{1}, x_{2}, \ldots, x_{N}\right)=0 \tag{16}
\end{equation*}
$$

where $H_{2, \ldots, N}$ is the concavification of the function $K(\mathbf{x})=\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right)$ with respect to the last $(N-1)$ variables.
Furthermore, there exists a continuous $N$-antisymmetric Hamiltonian $\bar{H}$ on $\Omega^{N}$, such that

$$
\begin{equation*}
L_{\bar{H}}\left(x, u_{1}(x), u_{2}(x), \ldots, u_{N-1}(x)\right)=\sum_{i=1}^{N-1}\left\langle u_{i}(x), x\right\rangle \text { for all } x \in \Omega \tag{17}
\end{equation*}
$$

2. Conversely, if $\left(u_{1}, \ldots, u_{N-1}\right)$ satisfy (15) for some $N$-sub-antisymmetric Hamiltonian $H$ that is concave in the first variable, convex in the other variables, then the $(N-1)$-tuple $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$ monotone.

Remark 3 Note that in the case $N=2, K(\mathbf{x})=H\left(x_{2}, x_{1}\right)$ is concave with respect to $x_{2}$, hence $H_{2}\left(x_{1}, x_{2}\right)=$ $H\left(x_{2}, x_{1}\right)$, and (16) becomes

$$
H\left(x_{1}, x_{2}\right)+H\left(x_{2}, x_{1}\right)=0
$$

thus $H$ is antisymmetric, recovering well-known results [9], [4], [7], [8].
We start with the following lemma.
Lemma 4 Assume the $(N-1)$-tuple of bounded vector fields $\left(u_{1}, \ldots, u_{N-1}\right)$ on $\Omega$ is jointly $N$-monotone. Let $f\left(x_{1}, \ldots, x_{n}\right):=\sum_{l=1}^{N-1}\left\langle u_{l}\left(x_{1}\right), x_{1}-x_{l+1}\right\rangle$ and consider the function $\tilde{f}\left(x_{1}, \ldots, x_{n}\right)$ to be the convexification of $f$ with respect to the first variable, that is

$$
\begin{equation*}
\tilde{f}\left(x_{1}, x_{2}, \ldots, x_{N}\right)=\inf \left\{\sum_{k=1}^{n} \lambda_{k} f\left(x_{1}^{k}, x_{2}, \ldots, x_{N}\right): n \in \mathbb{N}, \lambda_{k} \geq 0, \sum_{k=1}^{n} \lambda_{k}=1, \sum_{k=1}^{n} \lambda_{k} x_{1}^{k}=x_{1}\right\} \tag{18}
\end{equation*}
$$

Then, $\tilde{f}$ satisfies the following properties:

1. $f \geq \tilde{f}$ on $\Omega^{N}$;
2. $\tilde{f}$ is convex in the first variable and concave with respect to the other variables;
3. $\tilde{f}(x, x, \ldots, x)=0$ for each $x \in \Omega$,
4. $\tilde{f}$ satisfies

$$
\begin{equation*}
\sum_{i=0}^{N-1} \tilde{f}\left(\sigma^{i}\left(x_{1}, \ldots, x_{N}\right)\right) \geq 0 \text { on } \Omega^{N} \tag{19}
\end{equation*}
$$

Proof: Since the $(N-1)$-tuple $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone, it is easy to see that the function

$$
f\left(x_{1}, \ldots, x_{n}\right):=\sum_{l=1}^{N-1}\left\langle u_{l}\left(x_{1}\right), x_{1}-x_{l+1}\right\rangle
$$

is linear in the last $(N-1)$ variables, that $f(x, x, \ldots, x)=0$, and that

$$
\begin{equation*}
\sum_{i=0}^{N-1} f\left(\sigma^{i}\left(x_{1}, \ldots, x_{N}\right)\right) \geq 0 \text { on } \Omega^{N} \tag{20}
\end{equation*}
$$

It is also clear that $f \geq \tilde{f}$, that $\tilde{f}$ is convex with respect to the first variable $x_{1}$, and that it is concave with respect to the other variables $x_{2}, \ldots, x_{N}$, since $f$ itself is concave (actually linear) with respect to $x_{2}, \ldots, x_{n}$. We now show that $\tilde{f}$ satisfies (19).

For that, we fix $x_{1}, x_{2}, \ldots, x_{N}$ in $\Omega$ and consider $\left(x_{1}^{k}\right)_{k=1}^{n}$ in $\Omega$, and $\left(\lambda_{k}\right)_{k}$ in $\mathbb{R}$ such that $\lambda_{k} \geq 0$ such that $\sum_{k=1}^{n} \lambda_{k}=1$ and $\sum_{k=1}^{n} \lambda_{k} x_{1}^{k}=x_{1}$. For each $k$, we have

$$
f\left(x_{1}^{k}, x_{2}, \ldots, x_{N}\right)+f\left(x_{2}, \ldots, x_{N}, x_{1}^{k}\right)+\ldots+f\left(x_{N}, x_{1}^{k}, x_{2}, \ldots, x_{N-1}\right) \geq 0
$$

Multiplying by $\lambda_{k}$, summing over $k$, and using that $f$ is linear in the last $(N-1)$-variables, we have

$$
\sum_{k=1}^{n} \lambda_{k} f\left(x_{1}^{k}, x_{2}, \ldots, x_{N}\right)+f\left(x_{2}, \ldots, x_{N}, x_{1}\right)+\ldots+f\left(x_{N}, x_{1}, x_{2}, \ldots, x_{N-1}\right) \geq 0
$$

By taking the infimum, we obtain

$$
\tilde{f}\left(x_{1}, x_{2}, \ldots, x_{N}\right)+\sum_{i=1}^{N-1} f\left(\sigma^{i}\left(x_{1}, x_{2}, \ldots, x_{N}\right)\right) \geq 0
$$

Let now $n \in \mathbb{N}, \lambda_{k} \geq 0, x_{N}^{k} \in \Omega$ be such that $\sum_{k=1}^{n} \lambda_{k}=1$ and $\sum_{k=1}^{n} \lambda_{k} x_{2}^{k}=x_{2}$. We have for every $1 \leq k \leq n$,

$$
\tilde{f}\left(x_{1}, x_{2}^{k}, x_{3}, \ldots, x_{N}\right)+f\left(x_{2}^{k}, x_{3}, \ldots, x_{1}\right)+\ldots+f\left(x_{N}, x_{1}, x_{2}^{k}, x_{3}, \ldots, x_{N-1}\right) \geq 0
$$

Multiplying by $\lambda_{k}$, summing over $k$ and using that $\tilde{f}$ is convex in the first variable and $f$ is linear in the last ( $N-1$ )-variables, we obtain

$$
\begin{array}{r}
\tilde{f}\left(x_{1}, x_{2}, x_{3}, \ldots, x_{N}\right)+\sum_{k=1}^{n} \lambda_{k} f\left(x_{2}^{k}, x_{3}, \ldots, x_{1}\right)+\ldots+f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \\
\geq \sum_{k=1}^{n} \lambda_{k} \tilde{f}\left(x_{1}, x_{2}^{k}, x_{3}, \ldots, x_{N}\right)+\sum_{k=1}^{n} \lambda_{k} f\left(x_{2}^{k}, x_{3}, \ldots, x_{1}\right)+\ldots+\sum_{k=1}^{n} \lambda_{k} f\left(x_{N}, x_{1}, x_{2}^{k}, x_{3}, \ldots, x_{N-1}\right) \geq 0 .
\end{array}
$$

By taking the infimum over all possible such choices, we get

$$
\tilde{f}\left(x_{1}, x_{2}, x_{3}, \ldots, x_{N}\right)+\tilde{f}\left(x_{2}, x_{3}, \ldots, x_{1}\right)+\ldots+f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \geq 0
$$

By repeating this procedure with $x_{3}, \ldots, x_{N-1}$, we get

$$
\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^{i}\left(x_{1}, x_{2}, \ldots, x_{N}\right)\right)+f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \geq 0
$$

Finally, since

$$
f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \geq-\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^{i}\left(x_{1}, x_{2},, \ldots, x_{N}\right)\right)
$$

and since $\tilde{f}$ is concave in the last $(N-1)$ variables, we have for fixed $x_{1}, x_{2}, \ldots, x_{N-1}$, that the function

$$
x_{N} \rightarrow-\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^{i}\left(x_{1}, x_{2},, \ldots, x_{N}\right)\right)
$$

is a convex minorant of $x_{N} \rightarrow f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right)$. It follows that

$$
f\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \geq \tilde{f}\left(x_{N}, x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}\right) \geq-\sum_{i=0}^{N-2} \tilde{f}\left(\sigma^{i}\left(x_{1}, x_{2},, \ldots, x_{N}\right)\right)
$$

which finally implies that $\sum_{i=0}^{N-1} \tilde{f}\left(\sigma^{i}\left(x_{1}, x_{2}, \ldots, x_{N}\right)\right) \geq 0$.
This clearly implies that $\tilde{f}(x, x, \ldots, x) \geq 0$ for any $x \in \Omega$. On the other hand, since $\tilde{f}(x, x, \ldots, x) \leq$ $f(x, x, \ldots, x)=0$, we get that $\tilde{f}(x, x, \ldots, x)=0$ for all $x \in \Omega$.

Proof of Theorem 2: Assume the $(N-1)$-tuple of vector fields $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone on $\Omega$, and consider the function $f\left(x_{1}, \ldots, x_{n}\right):=\sum_{l=1}^{N-1}\left\langle u_{l}\left(x_{1}\right), x_{1}-x_{l+1}\right\rangle$ as well as its convexification with respect to the first variable $\tilde{f}\left(x_{1}, \ldots, x_{n}\right)$.

By Lemma [4, the function $\psi\left(x_{1}, \ldots, x_{n}\right):=-\tilde{f}\left(x_{1}, \ldots, x_{n}\right)$ satisfies the following properties
(i) $x_{1} \rightarrow \psi\left(x_{1}, \ldots, x_{n}\right)$ is concave;
(ii) $\left(x_{2}, x_{3}, \ldots, x_{N}\right) \rightarrow \psi\left(x_{1}, \ldots, x_{n}\right)$ is convex;
(iii) $\psi\left(x_{1}, \ldots, x_{n}\right) \geq-f\left(x_{1}, \ldots, x_{n}\right)=\sum_{l=1}^{N-1}\left\langle u_{l}\left(x_{1}\right), x_{l+1}-x_{1}\right\rangle$;
(iv) $\psi$ is $N$-sub-antisymmetric.

Consider now the family $\mathcal{H}$ of functions $H: \Omega^{N} \rightarrow \mathbb{R}$ such that

1. $H\left(x_{1}, x_{2}, \ldots, x_{N}\right) \geq \sum_{l=1}^{N-1}\left\langle u_{l}\left(x_{1}\right), x_{l+1}-x_{1}\right\rangle$ for every $N$-tuple $\left(x_{1}, \ldots, x_{N}\right)$ in $\Omega^{N}$;
2. $H$ is concave in the first variable;
3. $H$ is jointly convex in the last $(N-1)$ variables;
4. $H$ is $N$-sub-antisymmetric.

Note that $\mathcal{H} \neq \emptyset$ since $\psi$ belongs to $\mathcal{H}$. Moreover, by $N$-subsymmetry, any $H \in \mathcal{H}$ satisfies for all $\mathbf{x}=\left(x_{1}, \ldots, x_{N}\right) \in \Omega^{N}$,

$$
\begin{equation*}
H(\mathbf{x}) \leq-\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right) \leq-\sum_{i=1}^{N-1} \psi\left(\sigma^{i}(\mathbf{x})\right) \tag{21}
\end{equation*}
$$

This also yields that

$$
\begin{equation*}
\sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{\ell+1}-x_{1}\right\rangle \leq H(\mathbf{x}) \leq-\sum_{i=2}^{N} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{i}\right), x_{i}-x_{i+\ell}\right\rangle \tag{22}
\end{equation*}
$$

where we denote $x_{i+N}:=x_{i}$ for $i=1, \ldots, \ell$. This yields that $H(x, x, \ldots, x)=0$ for every $H \in \mathcal{H}$ and any $x \in \Omega$.

On the other hand, it is easy to see that every directed family $\left(H_{i}\right)_{i}$ in $\mathcal{H}$ has a supremum $H_{\infty} \in \mathcal{H}$, meaning that $\mathcal{H}$ is a Zorn family, and therefore it has a maximal element $H$.

Consider now the function

$$
\bar{H}(\mathbf{x})=\frac{(N-1) H(\mathbf{x})-\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right)}{N}
$$

and note that
(i) $\bar{H}$ is $N$-antisymmetric, since

$$
\bar{H}(\mathbf{x})=\frac{1}{N} \sum_{i=1}^{N-1}\left[H(\mathbf{x})-H\left(\sigma^{i}(\mathbf{x})\right)\right]
$$

and each $K_{i}(\mathbf{x}):=H(\mathbf{x})-H\left(\sigma^{i}(\mathbf{x})\right)$ is $N$-antisymmetric.
(ii) $\bar{H} \geq H$ on $\Omega^{N}$, since

$$
N[\bar{H}(\mathbf{x})-H(\mathbf{x})]=-\sum_{i=0}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right) \geq 0
$$

because $H$ itself is $N$-sub-antisymmetric.

The maximality of $H$ would have implied that $H=\bar{H}$ is $N$-antisymmetric if only $\bar{H}$ was jointly convex in the last $(N-1)$-variables, but since this is not necessarily the case, we consider for $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{N}\right)$, the function

$$
K\left(x_{1}, x_{2}, \ldots, x_{N}\right)=K(\mathbf{x}):=-\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right)
$$

which is already concave in the first variable $x_{1}$. Its convexification in the last $(N-1)$-variables, that is

$$
K^{2, \ldots, N}(\mathbf{x})=\inf \left\{\sum_{i=1}^{n} \lambda_{i} K\left(x_{1}, x_{2}^{i}, \ldots, x_{N}^{i}\right) ; \lambda_{i} \geq 0, \sum_{i=1}^{n} \lambda_{i}\left(x_{2}^{i}, \ldots, x_{N}^{i}, 1\right)=\left(x_{2}, \ldots, x_{N}, 1\right)\right\}
$$

is still concave in the first variable, but is now convex in the last $(N-1)$ variables. Moreover,

$$
\begin{equation*}
H \leq K^{2, \ldots, N} \leq K=-\sum_{i=1}^{N-1} H \circ \sigma^{i} \tag{23}
\end{equation*}
$$

Indeed, $K^{2, \ldots, N} \leq K$ from the definition of $K^{2, \ldots, N}$, while $H \leq K^{2, \ldots, N}$ because $H \leq K$ and $H$ is already convex in the last ( $N-1$ )-variables. It follows that

$$
H \leq \frac{(N-1) H+K^{2, \ldots, N}}{N} \leq \frac{(N-1) H+K}{N}=\frac{(N-1) H-\sum_{i=1}^{N-1} H \circ \sigma^{i}}{N}=\bar{H}
$$

The function $H^{\prime}=\frac{(N-1) H+K^{2, \ldots, N}}{N}$ belongs to the family $\mathcal{H}$ and therefore $H=H^{\prime}$ by the maximality of $H$. This finally yields that $H$ is $N$-sub-antisymmetric, that $H(x, x, x)=0$ for all $x \in \Omega$ and that

$$
H(\mathbf{x})+H_{2, \ldots, N}(\mathbf{x})=0 \text { for every } \mathbf{x} \in \Omega^{N}
$$

where $H_{2, \ldots, N}=-K^{2, \ldots, N}$, which for a fixed $x_{1}$, is nothing but the concavification of $\left(x_{2}, \ldots, x_{N}\right) \rightarrow$ $\sum_{i=1}^{N-1} H\left(\sigma^{i}\left(x_{1}, x_{2}, \ldots, x_{N}\right)\right)$.

Note now that since for any $x_{1}, \ldots, x_{N}$ in $\Omega$,

$$
\begin{equation*}
H\left(x_{1}, x_{2}, \ldots x_{N}\right) \geq \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{\ell+1}-x_{1}\right\rangle \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
H\left(x_{1}, x_{1}, \ldots, x_{1}\right)=0 \tag{25}
\end{equation*}
$$

we have

$$
\begin{equation*}
H\left(x_{1}, x_{2}, \ldots, x_{N}\right)-H\left(x_{1}, \ldots, x_{1}\right) \geq \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{\ell+1}-x_{1}\right\rangle \tag{26}
\end{equation*}
$$

Since $H$ is convex in the last $(N-1)$ variables, this means that for all $x \in \Omega$, we have

$$
\begin{equation*}
\left(u_{1}(x), u_{2}(x), \ldots, u_{N-1}(x)\right) \in \partial_{2, \ldots, N} H(x, x, \ldots, x) . \tag{27}
\end{equation*}
$$

as claimed in (15). Note that this also yield that

$$
L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right)+H(x, x, \ldots, x)=\sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x\right\rangle \text { for all } x \in \Omega
$$

In other words, $L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right)=\sum_{\ell=1}^{N-1} \int_{\Omega}\left\langle u_{\ell}(x), x\right\rangle$ for all $x \in \Omega$. As above, consider

$$
\bar{H}(\mathbf{x})=\frac{(N-1) H(\mathbf{x})-\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right)}{N}
$$

We have that $\bar{H} \in \mathcal{H}_{N}(\Omega)$ and $\bar{H} \geq H$, and therefore $L_{\bar{H}} \leq L_{H}$. On the other hand, we have for all $x \in \Omega$,

$$
L_{\bar{H}}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right)=L_{\bar{H}}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right)+\bar{H}(x, x, \ldots, x) \geq \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x\right\rangle
$$

To prove (17), we use the appendix in [6] to deduce that for $i=2, \ldots, N$, the gradients $\nabla_{i} H(x, x, \ldots, x)$ actually exist for a.e. $x$ in $\Omega$.

The converse is straightforward since if (27) holds, then (26) does, and since we also have (25), then the property that $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone follows from (24) and the sub-antisymmetry of $H$.

In the case of a single $N$-monotone vector field, we can obviously apply the above theorem to the ( $N-1$ )tuple $(u, 0, \ldots, 0)$ which is then $N$-monotone to find a $N$-sub-antisymmetric Hamiltonian $H$, which is concave in the first variable, convex in the last $(N-1)$ variables such that

$$
\begin{equation*}
(-u(x), u(x), 0, \ldots, 0)=\nabla H(x, x, \ldots, x) \quad \text { for a.e. } x \in \Omega \tag{28}
\end{equation*}
$$

However, in this case we can restrict ourselves to $N$-cyclically sub-antisymmetric functions of two variables and establish the following extension of the Theorem of Krauss.

Theorem 5 If $u$ is $N$-cyclically monotone on $\Omega$, then there exists a concave-convex function of two variables $F$ that is $N$-cyclically sub-antisymmetric, such that

$$
\begin{equation*}
(-u(x), u(x) \in \partial F(x, x) \text { for all } x \in \Omega \tag{29}
\end{equation*}
$$

where $\partial H$ is the sub-differential of $H$ as a concave-convex function [11]. Moreover,

$$
\begin{equation*}
u(x)=\nabla_{2} F(x, x) \text { for a.e. } x \in \Omega \tag{30}
\end{equation*}
$$

Proof: Let $f(x, y)=\langle u(x), x-y\rangle$ and let $f^{1}(x, y)$ be its convexification in $x$ for fixed $y$, that is

$$
\begin{equation*}
f^{1}(x, y)=\inf \left\{\sum_{k=1}^{n} \lambda_{k} f\left(x_{k}, y\right): \lambda_{k} \geq 0, \sum_{k=1}^{n} \lambda_{k}=1, \sum_{k=1}^{n} \lambda_{k} x_{k}=x\right\} \tag{31}
\end{equation*}
$$

Since $f(x, x)=0, f$ is linear in $y$, and $\sum_{i=1}^{N} f\left(x_{i}, x_{i+1}\right) \geq 0$ for any cyclic family $x_{1}, \ldots, x_{N}, x_{N+1}=x_{1}$ in $\Omega$, it is easy to show that $f \geq f^{1}$ on $\Omega, f^{1}$ is convex in the first variable and concave with respect to the second, $f^{1}(x, x)=0$ for each $x \in \Omega$, and that $f^{1}$ is $N$-cyclically supersymmetric in the sense that for any cyclic family $x_{1}, \ldots, x_{N}, x_{N+1}=x_{1}$ in $\Omega$, we have $\sum_{i=1}^{N} f^{1}\left(x_{i}, x_{i+1}\right) \geq 0$.

Consider now $F(x, y)=-f^{1}(x, y)$ and note that $x \rightarrow F(x, y)$ is concave, $y \rightarrow F(x, y)$ is convex, $F(x, y) \geq$ $-f(x, y)=\langle u(x), y-x\rangle$ and $F$ is N-cyclically sub-antisymmetric. By the antisymmetry, we have

$$
\begin{equation*}
\left\langle u\left(x_{1}\right), x_{2}-x_{1}\right\rangle \leq F\left(x_{1}, x_{2}\right) \leq\left\langle u\left(x_{2}\right), x_{2}-x_{1}\right\rangle \tag{32}
\end{equation*}
$$

which yields that $(-u(x), u(x)) \in \partial F(x, x)$ for all $x \in \Omega$.
Since $F$ is anti-symmetric and concave-convex, the possibly multivalued map $x \rightarrow \partial_{2} F(x, x)$ is monotone on $\Omega$, and therefore single-valued and differentiable almost everywhere [10]. This completes the proof.

Remark 6 Note that we cannot expect to have a function $F$ such that $\sum_{i=1}^{N} F\left(x_{i}, x_{i+1}\right)=0$ for all cyclic families $x_{1}, \ldots, x_{N}, x_{N+1}=x_{1}$ in $\Omega$. Actually, we believe that the only function satisfying such an $N$-antisymmetry for $N \geq 3$ must be of the form $F(x, y)=f(x)-f(y)$. This is the reason why one needs to consider functions of $N$-variables in order to get $N$-antisymmetry. In other words, the function defined by

$$
\begin{equation*}
H\left(x_{1}, x_{2}, \ldots, x_{N}\right):=\frac{(N-1) F\left(x_{1}, x_{2}\right)-\sum_{i=2}^{N-1} F\left(x_{i}, x_{i+1}\right)}{N} \tag{33}
\end{equation*}
$$

is $N$-antisymmetric in the sense of (6) and $H\left(x_{1}, x_{2} \ldots, x_{N}\right) \geq F\left(x_{1}, x_{2}\right)$ for all $\left(x_{1}, x_{2} \ldots, x_{N}\right)$ in $\Omega^{N}$.

## 3 Variational characterization of monotone vector fields

In order to simplify the exposition, we shall always assume in the sequel that $d \mu$ is Lebesgue measure $d x$ normalized to be a probability on $\Omega$. We shall also assume that $\Omega$ is convex and that its boundary has measure zero.

Theorem 7 Let $u_{1}, \ldots, u_{N-1}: \Omega \rightarrow \mathbb{R}^{d}$ be bounded measurable vector fields. The following properties are then equivalent:

1. The $(N-1)$-tuple $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone a.e., that is there exists a measure zero set $\Omega_{0}$ such that $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone on $\Omega \backslash \Omega_{0}$.
2. The infimum of the following Monge-Kantorovich problem

$$
\begin{equation*}
\left.\inf \left\{\int_{\Omega^{N}} \sum_{\ell=1}^{N-1}\left\langle u\left(x_{1}\right), x_{1}-x_{\ell+1}\right\rangle d \pi\left(x_{1}, x_{2}, \ldots, x_{N}\right)\right) ; \pi \in \mathcal{P}_{\mathrm{sym}}^{\mu}\left(\Omega^{N}\right)\right\} \tag{34}
\end{equation*}
$$

is equal to zero, and is therefore attained by the push-forward of $\mu$ by the map $x \rightarrow(x, x, \ldots, x)$.
3. $\left(u_{1}, \ldots, u_{N-1}\right)$ is in the polar of $\mathcal{S}_{N}(\Omega, \mu)$ in the following sense,

$$
\begin{equation*}
\inf \left\{\int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x-S^{\ell} x\right\rangle d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}=0 \tag{35}
\end{equation*}
$$

4. The following holds:

$$
\begin{equation*}
\inf \left\{\int_{\Omega} \sum_{\ell=1}^{N-1}\left|u_{\ell}(x)-S^{\ell} x\right|^{2} d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}=\sum_{\ell=1}^{N-1} \int_{\Omega}\left|u_{\ell}(x)-x\right|^{2} d \mu \tag{36}
\end{equation*}
$$

5. There exists a $N$-sub-antisymmetric Hamiltonian $H$ which is concave in the first variable, convex in the last $(N-1)$ variables such that

$$
\begin{equation*}
\left(u_{1}(x), \ldots, u_{N-1}(x)\right)=\nabla_{2, \ldots, N} H(x, x, \ldots, x) \quad \text { for a.e. } x \in \Omega . \tag{37}
\end{equation*}
$$

Moreover, $H$ is $N$-symmetric in the sense of (16).
6. The following duality holds:

$$
\inf \left\{\int_{\Omega} L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right) d \mu ; H \in \mathcal{H}_{N}(\Omega)\right\}=\sup \left\{\int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), S^{\ell} x\right\rangle d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}
$$

and the latter is attained at the identity map.
We start with the following lemma, which identifies those probabilities in $\mathcal{P}_{\mathrm{sym}}^{\mu}\left(\Omega^{N}\right)$ that are carried by graphs of functions from $\Omega$ to $\Omega^{N}$.

Lemma 8 Let $S: \Omega \rightarrow \Omega$ be a $\mu$-measurable map, then the following properties are equivalent:

1. The image of $\mu$ by the map $x \rightarrow\left(x, S x, \ldots, S^{N-1} x\right)$ belongs to $\mathcal{P}_{\mathrm{sym}}^{\mu}\left(\Omega^{N}\right)$.
2. $S$ is $\mu$-measure preserving and $S^{N}(x)=x \mu$-a.e.
3. For any bounded Borel measurable $N$-antisymmetric $H$ on $\Omega^{N}$, we have $\int_{\Omega} H\left(x, S x, \ldots, S^{N-1} x\right) d \mu=0$.

Proof. It is clear that 1) implies 3) since $\int_{\Omega^{N}} H(\mathbf{x}) d \pi(\mathbf{x})=0$ for any $N$-antisymmetric Hamiltonian $H$ and any $\pi \in \mathcal{P}_{\text {sym }}^{\mu}\left(\Omega^{N}\right)$.

That 2) implies 1) is also straightforward since if $\pi$ is the push-forward of $\mu$ by a map of the form $x \rightarrow\left(x, S x, \ldots, S^{N-1} x\right)$, where $S$ is a $\mu$-measure preserving $S$ with $S^{N} x=x \mu$ a.e. on $\Omega$, then for all $h \in L^{1}\left(\Omega^{N}, d \pi\right)$, we have

$$
\begin{aligned}
\int_{\Omega^{N}} h\left(x_{1}, \ldots, x_{N}\right) d \pi & =\int_{\Omega^{N}} h\left(x, S x, \ldots, S^{N-1} x\right) d \mu(x)=\int_{\Omega^{N}} h\left(S x, S^{2} x, \ldots, S^{N-1} x, S^{N} x\right) d \mu(x) \\
& =\int_{\Omega^{N}} h\left(S x, S^{2} x, \ldots, S^{N-1} x, x\right) d \mu(x)=\int_{\Omega^{N}} h\left(\sigma\left(x_{1}, \ldots, x_{N}\right)\right) d \pi
\end{aligned}
$$

We now prove that 2) and 3) are equivalent. Assuming first that $S$ is $\mu$-measure preserving such that $S^{N}=I$ $\mu$ a.e., then for every Borel bounded $N$-antisymmetric $H$, we have

$$
\begin{aligned}
\int_{\Omega} H\left(x, S x, S^{2} x, \ldots, S^{N-1} x\right) d \mu & =\int_{\Omega} H\left(S x, S^{2} x, \ldots, S^{N-1} x, x\right) d \mu \\
& =\ldots=\int_{\Omega} H\left(S^{N-1} x, x, S x, \ldots, S^{N-2} x\right) d \mu
\end{aligned}
$$

Since $H$ is $N$-antisymmetric, we can see that

$$
H\left(x, S x, \ldots, S^{N-1} x\right)+H\left(S x, S^{2} x, \ldots, S^{N-1} x, x\right)+\ldots H\left(S^{N-1} x, x, S x, . ., S^{N-2} x\right)=0
$$

It follows that $N \int_{\Omega} H\left(x, S x, S^{2} x, . ., S^{N-1} x\right) d \mu=0$.
For the reverse implication, assume $\int_{\Omega} H\left(x, S x, S^{2} x, \ldots, S^{N-1} x\right) d \mu=0$ for every $N$-antisymmetric Hamiltonian $H$. By testing this identity with the Hamiltonians

$$
H\left(x_{1}, x_{2}, \ldots, x_{N}\right)=f\left(x_{1}\right)-f\left(x_{i}\right),
$$

where $f$ is any continuous function on $\Omega$, one gets that $S$ is $\mu$-measure preserving. Now take the Hamiltonian

$$
H\left(x_{1}, x_{2}, \ldots, x_{N}\right)=\left|x_{1}-S x_{N}\right|-\left|S x_{1}-x_{2}\right|-\left|x_{2}-S x_{1}\right|+\left|S x_{2}-x_{3}\right|
$$

Note that $H \in \mathcal{H}_{N}(\Omega)$ since it is of the form $H\left(x_{1}, \ldots, x_{N}\right)=f\left(x_{1}, x_{2}, x_{N}\right)-f\left(x_{2}, x_{3}, x_{1}\right)$. Now test the above identity with such an $H$ to obtain

$$
0=\int_{\Omega} H\left(x, S x, S^{2} x, \ldots, S^{N-1} x\right) d \mu=\int_{\Omega}\left|x-S S^{N-1} x\right| d \mu
$$

It follows that $S^{N}=I \mu$ a.e. on $\omega$, and we are done.
Proof of Theorem 77: To show that (1) implies (2), it suffices to notice that if $\pi$ is a $\sigma$-invariant probability measure on $\Omega^{N}$ such that $\operatorname{proj}_{1} \pi=\mu$, then

$$
\begin{aligned}
\int_{\Omega^{N}} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{1}-x_{\ell+1}\right\rangle d \pi\left(x_{1}, \ldots, x_{N}\right) & =\frac{1}{N} \sum_{i=1}^{N} \int_{\Omega^{N}} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{i}\right), x_{i}-x_{i+\ell}\right\rangle d \pi\left(x_{1}, \ldots, x_{N}\right) \\
& =\frac{1}{N} \int_{\Omega^{N}}\left(\sum_{i=1}^{N} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{i}\right), x_{i}-x_{i+\ell}\right\rangle\right) d \pi\left(x_{1}, \ldots, x_{N}\right) \\
& \geq 0
\end{aligned}
$$

since $\left(u_{1}, \ldots, u_{N-1}\right)$ is jointly $N$-monotone. On the other hand, if $\pi$ is the $\sigma$-invariant measure obtained by taking the image of $\mu:=d x$ by $x \rightarrow(x, \ldots, x)$, then

$$
\int_{\Omega^{N}} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{1}-x_{\ell+1}\right\rangle d \pi\left(x_{1}, \ldots, x_{N}\right)=0 .
$$

To show that (2) implies (3), let $S$ be a $\mu$-measure preserving transformation on $\Omega$ such that $S^{N}=I \mu$ a.e. on $\Omega$. Then the image $\pi_{S}$ of $\mu$ by the map

$$
x \rightarrow\left(x, S x, S^{2} x, \ldots, S^{N-1} x\right)
$$

is $\sigma$-invariant, hence

$$
\int_{\Omega^{N}} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}\left(x_{1}\right), x_{1}-x_{\ell+1}\right\rangle d \pi_{S}\left(x_{1}, \ldots, x_{N}\right)=\int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x-S^{\ell} x\right\rangle d \mu \geq 0
$$

By taking $S=I$, we get that the infimum is necessarily zero.
The equivalence of (3) and (4) follows immediately from developing the square.

We now show that (3) implies (1). For that take $N$ points $x_{1}, x_{2}, \ldots, x_{N}$ in $\Omega$, and let $R>0$ be such that $B\left(x_{i}, R\right) \subset \Omega$. Consider the transformation

$$
S_{R}(x)=\left\{\begin{array}{c}
x-x_{1}+x_{2} \text { for } x \in B\left(x_{1}, R\right) \\
x-x_{2}+x_{3} \text { for } x \in B\left(x_{2}, R\right) \\
\ldots \\
x-x_{N}+x_{1} \text { for } x \in B\left(x_{N}, R\right) \\
x \text { otherwise }
\end{array}\right.
$$

It is easy to see that $S_{R}$ is a measure preserving transformation and that $S_{R}^{N}=I d$. We then have

$$
0 \leq \int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x-S_{R}^{\ell} x\right\rangle d \mu \leq \sum_{i=1}^{N} \int_{B\left(x_{i}, R\right)} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x_{i}-x_{\ell+i}\right\rangle d \mu
$$

Letting $R \rightarrow 0$, we get from Lebesgue's density theorem, that

$$
\frac{1}{\left|B\left(x_{i}, R\right)\right|} \int_{B\left(x_{i}, R\right)}\left\langle u_{\ell}(x), x_{i}-x_{\ell+i}\right\rangle d \mu \rightarrow\left\langle u_{\ell}\left(x_{i}\right), x_{i}-x_{\ell+i}\right\rangle,
$$

from which follows that $\left(u_{1}, \ldots, u_{N-1}\right)$ are jointly $N$-monotone a.e. on $\Omega$.
The fact that (1) is equivalent to (5) follows immediately from Theorem 2
To prove that 5) implies 6 ) note that for all $p_{i} \in \mathbb{R}^{d}, x \in \Omega, y_{i} \in \Omega, i=1, \ldots, N-1$,

$$
L_{H}\left(x, p_{1}, \ldots, p_{N-1}\right)+H\left(x, y_{1}, \ldots, y_{N-1}\right) \geq \sum_{i=1}^{N-1}\left\langle p_{i}, y_{i}\right\rangle
$$

which yields that for any $S \in \mathcal{S}_{N}(\Omega, \mu)$,

$$
\int_{\Omega}\left[L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right) d \mu+H\left(x, S x, \ldots, S^{N-1} x\right)\right] d \mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), S^{\ell} x\right\rangle d \mu
$$

If $H \in \mathcal{H}_{N}(\Omega)$ and $S \in \mathcal{S}_{N}(\Omega, \mu)$, we then have $\int_{\Omega} H\left(x, S x, \ldots, S^{N-1} x\right) d \mu=0$, and therefore

$$
\int_{\Omega} L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right) d \mu \geq \int_{\Omega} \sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), S^{\ell} x\right\rangle d \mu
$$

If now $H$ is the $N$-sub-antisymmetric Hamiltonian obtained by 5), which is concave in the first variable, convex in the last $(N-1)$ variables, then

$$
L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right)+H(x, x, \ldots, x)=\sum_{\ell=1}^{N-1}\left\langle u_{\ell}(x), x\right\rangle \text { for all } x \in \Omega \backslash \Omega_{0}
$$

and therefore $\int_{\Omega} L_{H}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right) d \mu=\sum_{\ell=1}^{N-1} \int_{\Omega}\left\langle u_{\ell}(x), x\right\rangle d \mu$.
Consider now

$$
\bar{H}(\mathbf{x})=\frac{(N-1) H(\mathbf{x})-\sum_{i=1}^{N-1} H\left(\sigma^{i}(\mathbf{x})\right)}{N}
$$

As before, we have that $\bar{H} \in \mathcal{H}_{N}(\Omega)$ and $\bar{H} \geq H$. Since $L_{\bar{H}} \leq L_{H}$, we have that $\int_{\Omega} L_{\bar{H}}\left(x, u_{1}(x), \ldots, u_{N-1}(x)\right) d \mu=$ $\sum_{\ell=1}^{N-1} \int_{\Omega}\left\langle u_{\ell}(x), x\right\rangle d \mu$ and (6) is proved.

Finally, note that (6) readily implies (3), which means that $\left(u_{1}, \ldots, u_{N-1}\right)$ is then jointly $N$-monotone.
We now consider again the case of a single $N$-cyclically monotone vector field.
Corollary 9 Let $u: \Omega \rightarrow \mathbb{R}^{d}$ be a bounded measurable vector field. The following properties are then equivalent:

1. $u$ is $N$-cyclically monotone a.e., that is there exists a measure zero set $\Omega_{0}$ such that $u$ is $N$-cyclically monotone on $\Omega \backslash \Omega_{0}$.
2. The infimum of the following Monge-Kantorovich problem

$$
\begin{equation*}
\inf \left\{\int_{\Omega^{N}}\left\langle u\left(x_{1}\right), x_{1}-x_{2}\right\rangle d \pi(\mathbf{x}) ; \pi \in \mathcal{P}_{\mathrm{sym}}^{\mu}\left(\Omega^{N}\right)\right\} \tag{38}
\end{equation*}
$$

is equal to zero, and is therefore attained by the push-forward of $\mu$ by the map $x \rightarrow(x, x, \ldots, x)$.
3. The vector field $u$ is in the polar of $\mathcal{S}_{N}(\Omega, \mu)$, that is

$$
\begin{equation*}
\inf \left\{\int_{\Omega}\langle u(x), x-S x\rangle d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}=0 \tag{39}
\end{equation*}
$$

4. The projection of $u$ on $\mathcal{S}_{N}(\Omega, \mu)$ is the identity map, that is

$$
\begin{equation*}
\inf \left\{\int_{\Omega}|u(x)-S x|^{2} d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}=\int_{\Omega}|u(x)-x|^{2} d \mu \tag{40}
\end{equation*}
$$

5. There exists a $N$-cyclically sub-antisymmetric function $H$ of two variables, which is concave in the first variable, convex in the second variable such that

$$
\begin{equation*}
u(x)=\nabla_{2} H(x, x) \quad \text { for a.e. } x \in \Omega . \tag{41}
\end{equation*}
$$

6. The following duality holds:

$$
\inf \left\{\int_{\Omega} L_{H}(x, u(x), 0, \ldots, 0) d \mu ; H \in \mathcal{H}_{N}(\Omega)\right\}=\sup \left\{\int_{\Omega}\langle u(x), S x\rangle d \mu ; S \in \mathcal{S}_{N}(\Omega, \mu)\right\}
$$

and the latter is attained at the identity map.
Proof: This is an immediate application of Theorem 7 applied to the $(N-1)$-tuplet vector fields $(u, 0, \ldots, 0)$, which is clearly jointly $N$-monotone on $\Omega \backslash \Omega_{0}$, whenever $u$ is $N$-monotone on $\Omega \backslash \Omega_{0}$.

Remark 10 Note that the sets of $\mu$-measure preserving $N$-involutions $\left(\mathcal{S}_{N}(\Omega, \mu)\right)_{N}$ do not form a nested family, that is $\mathcal{S}_{N}(\Omega, \mu)$ is not necessarily included in $\mathcal{S}_{M}(\Omega, \mu)$, whenever $N \leq M$, unless of course $M$ is a multiple of $N$. On the other hand, the above theorem shows that their polar sets, i.e.,

$$
\mathcal{S}_{N}(\Omega, \mu)^{0}=\left\{u \in L^{2}\left(\Omega, \mathbb{R}^{d}\right) ; \int_{\Omega}\langle u(x), x-S x\rangle d \mu \geq 0 \text { for all } S \in \mathcal{S}_{N}(\Omega, \mu)\right\}
$$

which coincide with the $N$-cyclically monotone maps, satisfy

$$
\mathcal{S}_{N+1}(\Omega, \mu)^{0} \subset \mathcal{S}_{N}(\Omega, \mu)^{0}
$$

for every $N \geq 1$. This can also be seen directly. Indeed, it is clear that a 2 -involution is a 4 -involution but not necessarily a 3 -involution. On the other hand, assume that $u$ is 3 -cyclically monotone operator, then for any transformation $S: \Omega \rightarrow \Omega$, we have

$$
\int_{\Omega}\langle u(x), x-S x\rangle d \mu+\int_{\Omega}\left\langle u(S x), S x-S^{2} x\right\rangle d \mu+\int_{\Omega}\left\langle u\left(S^{2} x\right), S^{2} x-x\right\rangle d \mu \geq 0 .
$$

If now $S$ is measure preserving, we have

$$
\int_{\Omega}\langle u(x), x-S x\rangle d \mu+\int_{\Omega}\langle u(x), x-S x\rangle d \mu+\int_{\Omega}\left\langle u\left(S^{2} x\right), S^{2} x-x\right\rangle d \mu \geq 0
$$

and if $S^{2}=I$, then $\int_{\Omega}\langle u(x), x-S x\rangle d \mu \geq 0$, which means that $u \in \mathcal{S}_{2}(\Omega, \mu)^{0}$. Similarly, one can show that any $(N+1)$-cyclically monotone operator belongs to $\mathcal{S}_{N}(\Omega, \mu)^{0}$. In other words, $\mathcal{S}_{N+1}(\Omega, \mu)^{0} \subset \mathcal{S}_{N}(\Omega, \mu)^{0}$ for all $N \geq 2$. Note that $\mathcal{S}_{1}(\Omega, \mu)^{0}=\{I\}^{0}=L^{2}\left(\Omega, \mathbb{R}^{d}\right)$, while

$$
\mathcal{S}(\Omega, \mu)^{0}=\cap_{N} \mathcal{S}_{N}(\Omega, \mu)^{0}=\left\{u \in L^{2}\left(\Omega, \mathbb{R}^{d}\right), u=\nabla \phi \text { for some convex function } \phi \text { in } W^{1,2}\left(\mathbb{R}^{d}\right)\right\}
$$

in view of classical results of Rockafellar (11] and Brenier [1].
Remark 11 In a forthcoming paper [6], the above result is extended to give a similar decomposition for any family of bounded measurable vector fields $u_{1}, u_{2}, \ldots, u_{N-1}$ on $\Omega$. It is shown there that there exists a measure preserving $N$-involution $S$ on $\Omega$ and an $N$-antisymmetric Hamiltonian $H$ on $\Omega^{N}$ such that for $i=1, \ldots, N-1$, we have

$$
u_{i}(x)=\nabla_{i+1} H\left(x, S x, S^{2} x, \ldots S^{N-1} x\right) \quad \text { for a.e. } x \in \Omega .
$$

## References

[1] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44 (1991), 375-417.
[2] S. P. Fitzpatrick, Representing monotone operators by convex functions, Proc. Centre for Math. Analysis 20 (1989), 59-65.
[3] W. Gangbo, An elementay proof of the polar factorization of vector-valued functions, Arch. Rat. Math. Analysis 128, No.5, (1994) 381-399.
[4] N. Ghoussoub, Selfdual partial differential systems and their variational principles, Springer Monograph in Mathematics, Springer-Verlag (2008), 356 p.
[5] N. Ghoussoub, B. Maurey, Remarks on multidimensional symmetric Monge-Kantorovich problems, Discrete and Continuous Dynamical Systems-A, special issue on "Optimal Transport and Applications" (2012)
[6] N. Ghoussoub, A. Moameni, Symmetric Monge-Kantorovich problems and polar decompositions of vector fields, Preprint (2012)
[7] N. Ghoussoub, A. Moameni, A Self-dual Polar Factorization for Vector Fields, Comm. Pure. Applied. Math (2012) 21pp
[8] P. Millien, A functional analytic approach to the selfdual polar decomposition, Master thesis, UBC (2011).
[9] E. Krauss, A representation of arbitrary maximal monotone operators via subgradients of skewsymmetric saddle functions, Nonlinear Anal. 9 (1985), no. 12, 1381-1399,
[10] R. R. Phelps, Convex functions, monotone operators and differentiability, Lecture Notes in Math. 1364, Springer Verlag, New York, Berlin, Tokyo, (1998), 2nd edition 1993.
[11] T. Rockafellar, Convex Analysis, 1970, Princeton University Press.


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