



**HAL**  
open science

# An Optimal Bandwidth For Difference-in-Difference Estimation with a Continuous Treatment and an Heterogeneous Adoption Design

Clément de Chaisemartin, Xavier d'Haultfoeuille

► **To cite this version:**

Clément de Chaisemartin, Xavier d'Haultfoeuille. An Optimal Bandwidth For Difference-in-Difference Estimation with a Continuous Treatment and an Heterogeneous Adoption Design. 2022. hal-03873937v1

**HAL Id: hal-03873937**

**<https://sciencespo.hal.science/hal-03873937v1>**

Preprint submitted on 27 Nov 2022 (v1), last revised 16 Jan 2025 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# An Optimal Bandwidth For Difference-in-Difference Estimation with a Continuous Treatment and an Heterogeneous Adoption Design

Clément de Chaisemartin      Xavier D’Haultfoeulle

November 23, 2022

**Setup** We consider a design with two time periods, such that  $D_1 = 0$  and  $D_2 > 0$ : no unit is treated at period 1, and all units receive a positive treatment dose at period 2. This corresponds to an heterogeneous adoption design, where all units adopt the treatment at period 2 but with varying treatment intensities. We assume that  $D_2$  is continuously distributed on  $\mathbb{R}_+$ . Accordingly,  $P(D_2 = 0) = 0$ : there are no stayers such that  $D_1 = D_2$ . On the other hand, we assume that there are quasi-stayers:  $f_{D_2}(0) > 0$  and  $d \mapsto f_{D_2}(d)$  is continuous, where  $f_{D_2}(\cdot)$  is the density of the period-two treatment. The potential outcome at period  $t$  under treatment  $d$  is  $Y_t(d)$ , and the observed outcome is  $Y_t := Y_t(D_t)$ .

**Target parameter** Our target parameter is

$$\theta_0 := E[Y_2(D_2) - Y_2(0)],$$

the average effect, at period 2, of switching the treatment from 0 to its actual value. We conjecture that the estimation strategy we propose below would still apply, up to a normalization, if one were to consider instead

$$\frac{E[Y_2(D_2) - Y_2(0)]}{E[D_2]} = E \left[ \frac{D_2}{E[D_2]} \times \frac{Y_2(D_2) - Y_2(0)}{D_2} \right],$$

a weighted average of the slopes of units’ potential outcomes functions between 0 and their actual treatments, where units with a larger period-two treatment receive more weight. On the other hand, we conjecture that the estimation strategy we propose below would not readily extend if one were to consider  $E[(Y_2(D_2) - Y_2(0))/D_2]$ , the unweighted average of the slopes of units’ potential outcomes functions between 0 and their actual treatments.

**Identifying assumption and estimation strategy** We assume strong exogeneity:

$$E[Y_2(0) - Y_1(0)|D_2] = E[Y_2(0) - Y_1(0)] =: \mu.$$

Then  $\theta_0 = E[Y_2 - Y_1] - \mu$ .  $\mu$  is for instance identified by  $E[Y_2 - Y_1|D_2 = 0]$ , thus implying that  $\theta_0$  is also identified. However, estimating  $E[Y_2 - Y_1|D_2 = 0]$  is not straightforward, as  $P(D_2 = 0) = 0$ : there are no stayers. Instead, we propose to use observations with  $D_2$  lower than some bandwidth  $h$  to estimate  $E[Y_2 - Y_1|D_2 = 0]$ . We derive below an optimal bandwidth  $h$ , namely a bandwidth minimizing an asymptotic approximation of the mean-squared error of the resulting estimator of  $\theta_0$ , in the spirit of the work of Imbens & Kalyanaraman (2012) for regression discontinuity designs.

**Optimal bandwidth to estimate  $\theta_0$**  We assume we have an iid sample  $(D_{2i}, Y_{1i}, Y_{2i})_{i=1, \dots, n}$ . We estimate  $\theta_0$  by:

$$\hat{\theta}_h = \frac{1}{n} \sum_{i=1}^n (Y_{i2} - Y_{i1}) - \hat{\mu}_h,$$

with  $\hat{\mu}_h$  the intercept in the local linear regression of  $Y_{i2} - Y_{i1}$  on  $D_{i2}$  for the  $i$ 's s.t.  $D_{i2} \leq h$ :

$$\hat{\mu}_h = \frac{1}{n_h} \sum_{i: D_{i2} \leq h} (Y_{i2} - Y_{i1}) - \frac{\overline{D_{2h}}}{\hat{V}_h(D_2)} \widehat{\text{Cov}}_h(Y_2 - Y_1, D_2),$$

where  $n_h = \#\{i : D_{i2} \leq h\}$ ,  $\overline{D_{2h}} = \sum_{i: D_{i2} \leq h} D_{i2} / n_h$  and

$$\begin{aligned} \hat{V}_h(D_2) &= \frac{1}{n_h} \sum_{i: D_{i2} \leq h} (D_{i2} - \overline{D_{2h}})^2, \\ \widehat{\text{Cov}}_h(Y_2 - Y_1, D_2) &= \frac{1}{n_h} \sum_{i: D_{i2} \leq h} (D_{i2} - \overline{D_{2h}})(Y_{i2} - Y_{i1}). \end{aligned}$$

We now derive asymptotic approximations of the bias and variance of  $\hat{\theta}_h$ , conditional on  $\mathbf{D} := (D_{12}, \dots, D_{n2})$ , when  $n \rightarrow +\infty$  and  $h \rightarrow 0$ . Let  $f(d) = E[Y_2 - Y_1|D_2 = d]$ , so that  $\mu = f(0)$  and assume that  $f$  is twice differentiable at 0. Then:

$$\begin{aligned} -\text{Bias}(\hat{\theta}_h|\mathbf{D}) &= \text{Bias}(\hat{\mu}_h|\mathbf{D}) \\ &= \frac{1}{n_h} \sum_{i: D_{i2} \leq h} f(D_{i2}) - \frac{\overline{D_{2h}}}{\hat{V}_h(D_2)} \frac{1}{n_h} \sum_{i: D_{i2} \leq h} f(D_{i2})(D_{i2} - \overline{D_{2h}}) - f(0) \\ &\simeq f(0) - f(0) + f'(0)\overline{D_{2h}} - f'(0) \frac{\overline{D_{2h}}}{\hat{V}_h(D_2)} \hat{V}_h(D_2) + \frac{f''(0)}{2} \overline{D_{2h}}^2 \\ &\quad - \frac{f''(0)\overline{D_{2h}}}{2\hat{V}_h(D_2)} \widehat{\text{Cov}}_h(D_{i2}^2, D_{i2}) \\ &= \frac{f''(0)}{2} \left[ \overline{D_{2h}}^2 - \frac{\overline{D_{2h}}}{\hat{V}_h(D_2)} \widehat{\text{Cov}}_h(D_{i2}^2, D_{i2}) \right], \end{aligned}$$

where the approximation follows from a Taylor expansion of order 2.

Now, remark that as  $h \rightarrow 0$ ,

$$\begin{aligned} E[D_2^k | D_2 \leq h] &= \frac{\int_0^h u^k f_{D_2}(u) du}{\int_0^h f_{D_2}(u) du} \\ &\sim \frac{f_{D_2}(0) \int_0^h u^k du}{f_{D_2}(0) \int_0^h du} \\ &\sim \frac{h^k}{k+1}, \end{aligned}$$

where the equivalence follows by continuity of  $f_{D_2}$  at 0. This implies that  $E[D_2^2 | D_2 \leq h] / E[D_2 | D_2 \leq h]^2 \rightarrow 4/3$  as  $h \rightarrow 0$ . Therefore, as  $h \rightarrow 0$ ,

$$\frac{E[D_2 | D_2 \leq h]^2}{V(D_2 | D_2 \leq h)} \rightarrow 3. \quad (1)$$

This implies after some algebra that

$$\text{Bias}(\hat{\theta}_h | \mathbf{D}) \sim \frac{f''(0)h^2}{12}.$$

Now, we consider the conditional variance of  $\hat{\theta}_h$ . Let  $\sigma^2(d) := V(Y_2 - Y_1 | D_2 = d)$ , assumed to be continuous at  $d = 0$ , and let  $\sigma^2 := E[\sigma^2(D_2)]$ . We have:

$$\begin{aligned} V(\hat{\theta}_h | \mathbf{D}) &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2(D_{2i}) + V(\hat{\mu}_h | \mathbf{D}) - \frac{2}{n} \text{Cov} \left( \sum_{i: D_{i2} \leq h} (Y_{i2} - Y_{i1}), \hat{\mu}_h | \mathbf{D} \right) \\ &= \frac{\sigma^2 + o_p(1)}{n} + \frac{V \left( \sum_{i: D_{i2} \leq h} Y_{i2} - Y_{i1} | \mathbf{D} \right)}{n_h} \left( \frac{1}{n_h} - \frac{2}{n} \right) + \left( \frac{\overline{D_{2h}}}{\hat{V}_h(D_2)} \right)^2 \frac{1}{n_h^2} \sum_{i: D_{i2} \leq h} (D_{i2} - \overline{D_{2h}})^2 \\ &\quad \times \sigma^2(D_{i2}) - \frac{2\overline{D_{2h}}}{n_h \hat{V}_h(D_2)} \left( \frac{1}{n_h} - \frac{1}{n} \right) \sum_{i: D_{i2} \leq h} (D_{2i} - \overline{D_{2h}}) \sigma^2(D_{2i}) \\ &= \frac{\sigma^2}{n} + (\sigma^2(0) + o(1)) \left( \frac{1}{n_h} - \frac{2}{n} \right) + \frac{\sigma^2(0) + o(1)}{n_h} \frac{(\overline{D_{2h}})^2}{\hat{V}_h(D_2)} + o_p \left( \frac{1}{n} \right) \\ &= \frac{4\sigma^2(0)}{nhf_{D_2}(0)} (1 + o_p(1)). \end{aligned}$$

The second equality follows from the law of large numbers. The third follows from Taylor expansions of order 0. The fourth follows from  $n_h/n = o_p(1)$ ,  $n_h = nP(D_2 \leq h)(1 + o_p(1)) = nhf_{D_2}(0)(1 + o_p(1))$  (where the last equality follows again by continuity of  $f_{D_2}$  at 0), and (1).

Hence,

$$\text{MSE}(\hat{\theta}_h | \mathbf{D}) \simeq \frac{4\sigma^2(0)}{nhf_{D_2}(0)} + \frac{f''(0)^2 h^4}{144}.$$

The optimal  $h^*$  of the approximation satisfies

$$0 = \frac{-4\sigma^2(0)}{nh^{*2}f_{D_2}(0)} + \frac{f''(0)^2h^{*3}}{36}.$$

Hence,

$$h^* = \left[ \frac{144\sigma^2(0)}{nf''(0)^2f_{D_2}(0)} \right]^{1/5}.$$

**Estimation of the optimal bandwidth** Estimating  $h^*$  requires estimating  $\sigma^2(0)$ ,  $f''(0)$ , and  $f_{D_2}(0)$ .  $\sigma^2(0)$  may be estimated as the difference between the intercept in a local linear regression of  $(Y_{i2} - Y_{i1})^2$  on  $D_{i2}$  and the square of the intercept in a local linear regression of  $Y_{i2} - Y_{i1}$  on  $D_{i2}$ .  $f''(0)$  may be estimated as the coefficient on  $D_{i2}^2$  in a local quadratic regression of  $Y_{i2} - Y_{i1}$  on  $D_{i2}$  and  $D_{i2}^2$ . Finally,  $f_{D_2}(0)$  may be estimated using a kernel density estimator.

**Next steps** This project is still at a very preliminary stage, and many important steps still need to be conducted. First, in lieu of the heuristic derivation of the optimal bandwidth above, a formal derivation will have to be provided. Second, the asymptotic distribution of  $n^{2/5}(\hat{\theta}_{h^*} - \theta_0)$  will have to be provided. This distribution will have a first-order bias, which will have to be estimated to construct valid confidence intervals, in the spirit of the work of Calonico et al. (2014) for regression discontinuity designs. Third, in regression discontinuity designs, the treatment effect estimator is the difference between two estimators converging at the non-parametric  $n^{2/5}$  rate. Here,  $\hat{\theta}_{h^*}$  is the difference between an estimator converging at the parametric rate and an estimator converging at the non-parametric rate. Therefore, the asymptotic distribution of  $\hat{\theta}_{h^*}$  only depends on that of the second estimator. This is slightly unusual, and may reduce the ability of this asymptotic distribution to replicate the finite sample distribution of  $\hat{\theta}_{h^*}$ , an issue that we shall investigate via simulations.

## References

- Calonico, S., Cattaneo, M. D. & Titiunik, R. (2014), ‘Robust nonparametric confidence intervals for regression-discontinuity designs’, *Econometrica* **82**(6), 2295–2326.
- Imbens, G. & Kalyanaraman, K. (2012), ‘Optimal bandwidth choice for the regression discontinuity estimator’, *The Review of Economic Studies* **79**(3), 933–959.  
**URL:** <http://dx.doi.org/10.1093/restud/rdr043>