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# Identification of Hedonic Equilibrium and Nonseparable Simultaneous Equations

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This paper derives conditions under which preferences and technology are nonparametrically identified in hedonic equilibrium models. With products differentiated along a quality index and agents characterized

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by scalar unobserved heterogeneity, single-crossing conditions on preferences and technology provide identifying restrictions in previous work. We develop similar shape restrictions in the multiattribute case. These shape restrictions, based on optimal transport theory and generalized convexity, allow us to identify preferences for goods differentiated along multiple dimensions from the observation of a single market. We thereby derive identification results for nonseparable simultaneous equations and multiattribute hedonic equilibrium models with (possibly) multiple dimensions of unobserved heterogeneity. One of our results is a proof of absolute continuity of the distribution of endogenously traded qualities, which is of independent interest.

## Introduction

Hedonic models were initially introduced by Court (1939) to price highly differentiated goods in terms of their attributes. The vast subsequent literature on hedonic regressions of prices on attributes aimed at measuring the marginal willingness of consumers to pay for the attributes of the good they acquired, or the marginal willingness of workers to accept compensation for the attributes of their occupations. When unobservable taste for attributes drives the consumers' choices, however, a simple regression of price on attributes cannot inform us on the willingness to pay for quality levels different from the ones characterizing the good actually acquired. Nor can they inform us on the willingness to pay for characteristics of the good they would acquire under counterfactual market conditions, with different endowments, preferences, and technology.

The willingness to pay for counterfactual transactions, together with structural parameters of preferences and technology, can be recovered with a general equilibrium theory of hedonic models, dating back to Tinbergen (1956) and Rosen (1974); see Heckman (2019) for an account of their respective contributions. The common underlying framework, which we also adopt here, is that of a perfectly competitive market with heterogeneous buyers and sellers and traded product quality bundles and prices that arise endogenously in equilibrium.<sup>1</sup> Rosen (1974) proposes a two-step procedure to estimate general hedonic models and thereby analyze general equilibrium effects of changes in buyer-seller compositions, preferences, and technology on qualities traded at equilibrium and their price (see Heckman 1999). The first step is a regression of prices on attributes,

<sup>1</sup> When preferences are quasi-linear in price and under mild semicontinuity assumptions, Chiappori, McCann, and Nesheim (2010) and Ekeland (2010) show that equilibria exist, in the form of a joint distribution of product and consumer types (who consumes what), a joint distribution of product and producer types (who produces what), and a price schedule such that markets clear for each traded product.

and the second is a simultaneous-equations estimation of the demand-and-supply system with marginal prices estimated in the first step as endogenous variables.

Brown and Rosen (1982) and Brown (1983) point out that changes in consumers' unobserved taste for attributes would lead them to source goods from different suppliers, so that exclusion restrictions from the supply side cannot be justified.<sup>2</sup> The literature on recovering marginal willingness to pay for counterfactual transactions has since followed three strategies: relying on multiple markets across space or time (see Brown and Rosen 1982, Brown 1983, Kahn and Lang 1988, Tauchen and Witte 2001, and many references in Kuminoff, Smith, and Timmins 2013), relying on specific functional forms for utility (Bajari and Benkard 2005, Bishop and Timmins 2011, 2019, and references therein), or assuming that consumers care about a single dimension of good heterogeneity, via a quality index (Ekeland, Heckman, and Nesheim 2002a, 2002b, 2004; Heckman, Matzkin, and Nesheim 2003, 2010; Epple, Peress, and Sieg 2010; Epple, Quintero, and Sieg 2020). Each of these strategies has drawbacks. Multimarket strategies rely on the assumption of no leakage between markets, or consumption substitution across time and space. They also rely on the assumption that preferences and the distribution of preference types are stable across markets, so that variation comes from the supply side and preferences are identified, but not technology (or vice versa with symmetric assumptions; see Ekeland, Heckman, and Nesheim 2004). Identification strategies based on specific parameterizations of preferences and technology cannot distinguish features of the specification that are crucial to identification and features that are convenient approximations. Identification proofs must also be repeated for each new parameterization, the suitability of which depends on the application (see the discussion in Yinger 2014). Finally, it is important in many applications to account for heterogeneity in consumers' (or workers') relative valuations of different attributes and hence move beyond the case of a scalar index of attributes.

This paper proposes an identification strategy based on a single market, where agents have heterogeneous relative valuations for different attributes, without relying on a specific parametric specification of preferences and technology. A leading case in the class of specifications we entertain is  $U(x, \varepsilon, z) = \bar{U}(x, z) + z'\varepsilon$ , where  $U$  is the valuation of the bundle of attributes  $z$  as a function of the vectors of observable and unobservable consumer characteristics  $x$  and  $\varepsilon$ , respectively. We show that for each choice of distribution of types  $\varepsilon$ , the function  $\bar{U}$  is recovered nonparametrically (and the same result holds for the supply side). Our contribution is a direct generalization of the main identification strategy in Heckman, Matzkin,

<sup>2</sup> See also Bartik (1987), Epple (1987), and Kahn and Lang (1988).

and Nesheim (2010). In that work, under a single-crossing condition<sup>3</sup> on the utility function, the first-order condition of the consumer problem yields an increasing demand function, that is, quality demanded by the consumer as an increasing function of her unobserved type, interpreted as unobserved taste for quality. Assortative matching guarantees uniqueness of demand, as the unique increasing function that maps the distribution of unobserved taste for quality, which is specified a priori, and the distribution of qualities, which is observed. Hence, demand is identified as a quantile function, as in Matzkin (2003). Identification, therefore, is driven by a shape restriction on the utility function.

The main achievement of this paper is to show that a suitable multivariate extension (called a “twist condition”) of the single-crossing shape restriction delivers the same identification result in hedonic equilibrium with multiple good quality dimensions. Heuristically, the proof mirrors Heckman, Matzkin, and Nesheim (2010) in that it first involves showing identification of inverse demand, which then allows the identification of marginal utility from the first-order condition of the consumer’s program. The identification of inverse demand, that is, a single-valued mapping from a vector of good qualities to a vector of unobserved consumer type, involves the twist shape restriction and cyclical monotonicity of the hedonic equilibrium solution. The recovery of marginal utility from the first-order condition of the consumer’s program is involved, because differentiability of the hedonic equilibrium price function is not guaranteed. Known conditions for differentiability of transport potentials in general optimal transport problems, due to Ma, Trudinger, and Wang (2005), which would yield differentiability of the hedonic equilibrium price in our context, are very strong and rule out many simple forms of the matching surplus (see chap. 12 of Villani 2009). We are able to bypass the Ma, Trudinger, and Wang (2005) conditions, using the special structure of the hedonic equilibrium, and to show approximate differentiability (definition 10.2 of Villani 2009, 218) of the price function, for which we need absolute continuity of the distribution of good qualities traded at equilibrium. To that end, we provide a set of mild conditions on the primitives under which the endogenous distribution of qualities traded at equilibrium is absolutely continuous. The proof of absolute continuity of the distribution of qualities traded at equilibrium is based on an argument from Figalli and Juillet (2008), also applied in Kim and Pass (2017).<sup>4</sup>

An important special case of our main identification theorem is the case where the consumer’s utility depends on consumer unobserved heterogeneity  $\varepsilon$  only through the index  $z'\varepsilon$ , where  $z$  is the vector of good

<sup>3</sup> Also known as a Spence-Mirlees or supermodularity condition.

<sup>4</sup> Figalli and Juillet (2008) and Kim and Pass (2017) focus on quadratic distance cost, i.e.,  $\zeta(x, \varepsilon, z) = -d^2(z, \varepsilon)$  in the notation of assumption H, and work in more exotic geometric spaces.

qualities. This case has the appealing interpretation that each dimension of unobservable taste is associated with a good quality dimension and the appealing feature that marginal utility is characterized as the solution of a convex program. However, choosing the dimension of unobserved consumer heterogeneity to be equal to the dimension of the vector of good qualities is a somewhat arbitrary modeling choice, and we provide an extension of our main identification theorem to cases where the dimension of unobserved consumer heterogeneity is lower, including a model with scalar unobserved heterogeneity. We derive a local identification result under mild conditions, but for global identification, we need a shape restriction on the endogenous price function, for which we know of no sufficient conditions on primitives. Another restrictive aspect of our main result is the necessary normalization of the distribution of unobserved heterogeneity when identifying primitives from a single market. We provide some relaxation of this constraint when data from multiple markets are available, but the results are still fragmentary.

The analysis of identification of inverse demand in hedonic equilibrium reveals that inverse demand satisfies a multivariate notion of monotonicity, whose definition depends on the form of the utility function that is maximized. In the univariate case, this notion reduces to monotonicity of inverse demand. In the case where the consumer's utility depends on consumer unobserved heterogeneity  $\varepsilon$  only through the index  $z'\varepsilon$ , where  $z$  is the vector of good qualities, inverse demand is the gradient of a convex function. We show that this notion of multivariate monotonicity is a suitable shape restriction to identify nonseparable-simultaneous-equations models, generalizing the quantile identification method of Matzkin (2003) and complementing results in Matzkin (2015), where monotonicity is imposed equation by equation.<sup>5</sup>

*Closely related work.*—On the identification of multiattribute hedonic equilibrium models, Ekeland, Heckman, and Nesheim (2004) require marginal utility (marginal product) to be additively separable in unobserved consumer (producer) characteristics. Heckman, Matzkin, and Nesheim (2010) show that demand is nonparametrically identified under a single-crossing condition and that various additional shape restrictions allow identification of preferences without additive separability (see also Heckman, Matzkin, and Nesheim 2003, 2005). Ekeland, Heckman, and Nesheim (2004) emphasize the one-dimensional case but argue that their results can be extended to multivariate attributes under a separability assumption in the utility (without restricting the distribution of unobserved heterogeneity). Our paper directly follows Heckman, Matzkin, and Nesheim (2010) and generalizes the insight therein to allow for heterogeneity in response to

<sup>5</sup> Not all results in Matzkin (2003) and Heckman, Matzkin, and Nesheim (2010) require normalization of the distribution of unobserved heterogeneity, as we do here.

different dimensions of amenities or qualities. However, the analyses in Heckman, Matzkin, and Nesheim (2010) and ours are nonnested. Heckman, Matzkin, and Nesheim (2010) consider several specifications with Barten scales that are outside the scope of our generalization. Nesheim's (2015; developed independently and concurrently) is the most closely related paper and complements our work. He achieves identification under an additive separability restriction, but without restricting the distribution of unobserved heterogeneity. He also imposes conditions from Ma, Trudinger, and Wang (2005) to obtain differentiability of the price. Chiappori, McCann, and Nesheim (2010) derive a matching formulation of hedonic models and thereby highlight the close relation between empirical strategies in matching markets and in hedonic markets. Chiappori, McCann, and Pass (2016) have a section on identification (posterior to this paper), where they use a similar strategy. Two main differences arise from the difference between matching and hedonic models. On the one hand, Chiappori, McCann, and Pass (2016) need an extra step to account for the fact that in their matching model the price function is not observed. On the other hand, they do not have to worry about regularity of endogenous objects such as the price and distribution of goods in the hedonic equilibrium setting. Galichon and Salanié (2012) extend the work of Choo and Siow (2006) and identify preferences in marriage markets, where agents match on discrete characteristics, as the unique solution of an optimal transport problem, but unlike the present paper, they are restricted to the case with a discrete quality space. The strategy is extended to the set-valued case by Chiong, Galichon, and Shum (2013), who use subdifferential calculus to identify dynamic discrete-choice problems. Dupuy, Galichon, and Henry (2014) use network flow techniques to identify discrete hedonic models.

On the identification of nonlinear simultaneous-equations models, Matzkin (2015) uses equation-by-equation monotonicity in the one-dimensional unobservables and exclusion restrictions. Berry and Haile (2018) consider transformation models, as in Matzkin (2008; see also Matzkin 2013). These strategies do not require normalization of the distribution of unobserved heterogeneity. Shi, Shum, and Song (2018) also (independently and in a very different context) use cyclical monotonicity for identification in panel discrete-choice models. Ekeland, Galichon, and Henry (2012) and Galichon and Henry (2012) propose a notion of multivariate quantile based on Brenier's theorem. Carrier, Chernozhukov, and Galichon (2016), coetaneous with the present paper, propose a conditional version of the optimal transport quantiles of Ekeland, Galichon, and Henry (2012) and Galichon and Henry (2012) and apply it to quantile regression, whereas Chernozhukov et al. (2017), also coetaneous with this paper, apply optimal transport quantiles to the definition of statistical depth, ranks, and signs. However, these papers do not consider identification. The present work is, to the best of our knowledge, the first to apply the notion of multivariate quantiles

based on optimal transport results to the identification of simultaneous equations, thus providing a multivariate extension of Matzkin's (2003) quantile identification idea.

*Organization of the paper.*—The remainder of the paper is organized as follows. Section I sets the hedonic equilibrium framework out. Section II gives a brief account of the methodology and main results on nonparametric identification of preferences in single-attribute hedonic models, mostly drawn from Ekeland, Heckman, and Nesheim (2004) and Heckman, Matzkin, and Nesheim (2010). Section III shows how these results and the shape restrictions that drive them can be extended to the case of multiple-attribute hedonic equilibrium markets. Section IV derives multivariate shape restrictions to identify nonseparable-simultaneous-equations models. Section V concludes. Proofs of the main results are relegated to the appendix, as are necessary background definitions and results on optimal transport theory and a list of our notational conventions.

## I. Hedonic Equilibrium and the Identification Problem

We consider a competitive environment, where consumers and producers trade a good or contract, fully characterized by its type or quality  $z$ . The set of feasible qualities  $Z \subseteq \mathbb{R}^d$  is assumed compact and given a priori, but the distribution of the qualities actually traded arises endogenously in the hedonic market equilibrium, as does their price schedule  $p(z)$ . Producers are characterized by their type  $\tilde{y} \in \tilde{Y} \subseteq \mathbb{R}^d$  and consumers by their type  $\tilde{x} \in \tilde{X} \subseteq \mathbb{R}^d$ . Type distributions  $P_{\tilde{x}}$  on  $\tilde{X}$  and  $P_{\tilde{y}}$  on  $\tilde{Y}$  are given exogenously, so that entry and exit are not modeled. Consumers and producers are price takers and maximize quasi-linear utility  $U(\tilde{x}, z) - p(z)$  and profit  $p(z) - C(\tilde{y}, z)$ , respectively. Utility  $U(\tilde{x}, z)$  (cost  $C(\tilde{y}, z)$ ) is upper (lower) semicontinuous and bounded. In addition, the set of qualities  $Z(\tilde{x}, \tilde{y})$  that maximize the joint surplus  $U(\tilde{x}, z) - C(\tilde{y}, z)$  for each pair of types  $(\tilde{x}, \tilde{y})$  is assumed to have a measurable selection. Then, Ekeland (2010) and Chiappori, McCann, and Nesheim (2010) show that an equilibrium exists in this market, in the form of a price function  $p$  on  $Z$ , a joint distribution  $P_{\tilde{x}z}$  on  $\tilde{X} \times Z$  and  $P_{\tilde{y}z}$  on  $\tilde{Y} \times Z$  such that their marginals on  $Z$  coincide, so that the market clears for each traded quality  $z \in Z$ . Uniqueness is not guaranteed; in particular, prices are not uniquely defined for nontraded qualities in equilibrium. Purity is not guaranteed either: an equilibrium specifies a conditional distribution  $P_{z|\tilde{x}}$  ( $P_{z|\tilde{y}}$ ) of qualities consumed by type  $\tilde{x}$  consumers (produced by type  $\tilde{y}$  producers). The quality traded by a given producer-consumer pair  $(\tilde{x}, \tilde{y})$  is not uniquely determined at equilibrium without additional assumptions.

Ekeland (2010) and Chiappori, McCann, and Nesheim (2010) further show that a pure equilibrium exists and is unique, under the following



additional assumptions. First, type distributions  $P_{\tilde{x}}$  and  $P_{\tilde{y}}$  are absolutely continuous. Second, gradients of utility and cost,  $\nabla_{\tilde{x}}U(\tilde{x}, z)$  and  $\nabla_{\tilde{y}}C(\tilde{y}, z)$ , respectively, exist and are injective as functions of quality  $z$ . The latter condition, also known as the *twist condition* in the optimal transport literature, ensures that all consumers of a given type  $\tilde{x}$  (all producers of a given type  $\tilde{y}$ ) consume (produce) the same quality  $z$  at equilibrium.

The identification problem consists in the recovery of structural features of preferences and technology from observation of traded qualities and their prices in a single market. The solution concept we impose in our identification analysis is the following feature of hedonic equilibrium, that is, maximization of surplus generated by a trade.

ASSUMPTION EC (Equilibrium concept). The joint distribution  $\gamma$  of  $(\tilde{X}, Z, \tilde{Y})$  and the price function  $p$  form a hedonic equilibrium; that is, they satisfy the following. The joint distribution  $\gamma$  has marginals  $P_{\tilde{x}}$  and  $P_{\tilde{y}}$ , and for  $\gamma$  almost all  $(\tilde{x}, z, \tilde{y})$ ,

$$U(\tilde{x}, z) - p(z) = \max_{z' \in Z} (U(\tilde{x}, z') - p(z')),$$

$$p(z) - C(\tilde{y}, z) = \max_{z' \in Z} (p(z') - C(\tilde{y}, z')).$$

In addition, observed qualities  $z \in Z(\tilde{x}, \tilde{y})$ , maximizing joint surplus  $U(\tilde{x}, z) - C(\tilde{y}, z)$  for each  $\tilde{x} \in \tilde{X}$  and  $\tilde{y} \in \tilde{Y}$ , lie in the interior of the set of feasible qualities  $Z$ , and  $Z(\tilde{x}, \tilde{y})$  is assumed to have a measurable selection. The joint surplus  $U(\tilde{x}, z) - C(\tilde{y}, z)$  is finite everywhere. We assume full participation in the market.<sup>6</sup>

Given observability of prices and the fact that producer type  $\tilde{y}$  (consumer type  $\tilde{x}$ ) does not enter into the utility function  $U(\tilde{x}, z)$  (cost function  $C(\tilde{y}, z)$ ) directly, we may consider the consumer and producer problems separately and symmetrically (see Ekeland, Heckman, and Nesheim 2002a). We focus on the consumer problem and on identification of utility function  $U(\tilde{x}, z)$ . Under assumptions ensuring purity and uniqueness of equilibrium, the model predicts a deterministic choice of quality  $z$  for a given consumer type  $\tilde{x}$ . We do not impose such assumptions, but we need to account for heterogeneity in consumption patterns even in case of a unique and pure equilibrium. Hence, we assume, as is customary, that consumer types  $\tilde{x}$  are only partially observable to the analyst. We write  $\tilde{x} = (x, \varepsilon)$ , where  $x \in X \subseteq \mathbb{R}^d$  is the observable part of the type vector and  $\varepsilon \in \mathbb{R}^d$  is the unobservable part. We make a separability assumption that will allow us to specify constraints on the interaction between consumer

<sup>6</sup> The possibility of nonparticipation can be modeled by adding isolated points to the sets of types and renormalizing distributions accordingly (see sec. 2.1 of Chiappori, McCann, and Nesheim 2010 for details).

unobservable type  $\varepsilon$  and good quality  $z$  in order to identify interactions between observable type  $x$  and good quality  $z$ .

ASSUMPTION H (Unobservable heterogeneity). Consumer type  $\tilde{x}$  is composed of observable type  $x$  with distribution  $P_x$  on  $X \subseteq \mathbb{R}^{d_x}$  and unobservable type  $\varepsilon$  with a priori specified conditional distribution  $P_{\varepsilon|x}$  on  $\mathbb{R}^d$ , with  $d_\varepsilon \leq d_z$ . The utility of consumers can be decomposed as  $U(\tilde{x}, z) = \bar{U}(x, z) + \zeta(x, \varepsilon, z)$ , where the functional form of  $\zeta$  is known but that of  $\bar{U}$  is not.<sup>7</sup>

The main primitive object of interest is the deterministic component of utility  $\bar{U}(x, z)$ . For convenience, we use the transformation  $V(x, z) := p(z) - \bar{U}(x, z)$ . This is called the consumer's *potential*, in line with the optimal transport terminology. Since the price is assumed to be identified, identification of  $V$  is equivalent to identification of  $\bar{U}$ . To achieve identification, we require fixing a choice of function  $\zeta$ : a leading example, discussed in section III.C.1, is the case  $\zeta(x, \varepsilon, z) = z\varepsilon$ . Identification also requires fixing the conditional distribution  $P_{\varepsilon|x}$  of unobserved heterogeneity. This corresponds to the normalization of the distribution of scalar unobservable utility in existing quantile identification strategies. Discrete-choice models also typically rely on a fixed distribution for unobservable heterogeneity (generally extreme valued). The requirements to fix both  $\zeta$  and  $P_{\varepsilon|x}$  is relaxed to some extent in section III.D.2, which entertains the possibility of further identification power using information from multiple markets.

## II. Single-Market Identification with Scalar Attribute

In this section, we recall and reformulate results of Heckman, Matzkin, and Nesheim (2010) on identification of single-attribute hedonic models. Suppose, for the purpose of this section, that  $d_x = d_\varepsilon = 1$ , so that unobserved heterogeneity is scalar, as is the quality dimension. Suppose also that  $\zeta$  is twice continuously differentiable in  $z$  and  $\varepsilon$ .

ASSUMPTION R (Regularity of preferences and technology). The functions  $\bar{U}(x, z)$ ,  $C(\tilde{y}, z)$ , and  $\zeta(x, \varepsilon, z)$  are twice continuously differentiable with respect to  $\varepsilon$  and  $z$ .

Suppose further (for ease of exposition) that  $V$  is twice continuously differentiable in  $z$ . The main identifying assumption is a shape restriction on utility called single-crossing, Spence-Mirlees, or supermodularity, depending on the context.

ASSUMPTION S1 (Spence-Mirlees). We have  $d_x = 1$  and  $\zeta_{\varepsilon z}(x, \varepsilon, z) > 0$  for all  $(x, \varepsilon, z)$ .

<sup>7</sup> Despite the notation used,  $\bar{U}$  should not necessarily be interpreted as "mean utility," since we allow for a general choice of  $\zeta$  and  $P_{\varepsilon|x}$ . If this interpretation is desirable in a particular application,  $\zeta$  and  $P_{\varepsilon|x}$  can be chosen in such a way that  $\mathbb{E}[\zeta(x, \varepsilon, z)|x] = 0$ .

The first-order condition of the consumer problem yields

$$\zeta_z(x, \varepsilon, z) = V_z(x, z), \tag{1}$$

which, under assumption S1, implicitly defines an inverse-demand function  $z \mapsto \varepsilon(x, z)$ , which specifies which unobserved type consumes quality  $z$ . Combining the second-order condition  $\zeta_{zz}(x, \varepsilon, z) < V_{zz}(x, z)$  and further differentiation of equation (1), that is,  $\zeta_{zz}(x, \varepsilon, z) + \zeta_{\varepsilon z}(x, \varepsilon, z)\varepsilon_z(x, z) = V_{zz}(x, z)$ , yields

$$\varepsilon_z(x, z) = \frac{V_{zz}(x, z) - \zeta_{zz}(x, \varepsilon, z)}{\zeta_{\varepsilon z}(x, \varepsilon, z)} > 0.$$

Hence, the inverse demand is increasing and is therefore identified as the unique increasing function that maps the distribution  $P_{z|x}$  to the distribution  $P_{\varepsilon|x}$ , namely, the quantile transform. Denoting  $F$  the cumulative distribution function corresponding to the distribution  $P$ , we therefore have identification of inverse demand according to the strategy put forward in Matzkin (2003) as

$$\varepsilon(x, z) = F_{\varepsilon|x}^{-1}(F_{z|x}(z|x)).$$

The single-crossing condition of assumption S1 on the consumer surplus function  $\zeta(x, \varepsilon, z)$  yields positive assortative matching, as in the Becker (1973) classical model. Consumers with higher taste for quality  $\varepsilon$  will choose higher qualities in equilibrium, and positive assortative matching drives identification of demand for quality. The important feature of assumption S1 is injectivity of  $\zeta_z(x, \varepsilon, z)$  relative to  $\varepsilon$ , and a similar argument would have carried through under  $\zeta_{zz}(x, \varepsilon, z) < 0$ , yielding negative assortative matching instead.

Once inverse demand is identified, the consumer potential  $V(x, z)$ , and hence the utility function  $\bar{U}(x, z)$ , can be recovered up to a constant by integration of the first-order condition (1):

$$\bar{U}(x, z) = p(z) - V(x, z) = p(z) - \int_0^z \zeta_z(x, \varepsilon(x, z'), z') dz'.$$

We summarize the previous discussion in the following identification statement, originally due to Heckman, Matzkin, and Nesheim (2010).

**PROPOSITION 1.** Under assumptions EC, H, R, and S1,  $\bar{U}_z(x, z)$  is non-parametrically identified, in the sense that  $z \mapsto \bar{U}_z(x, z)$  is the only marginal utility function compatible with the pair  $(P_{xz}, p)$ ; that is, any other marginal utility function coincides with it,  $P_{z|x}$  almost surely.

Unlike the demand function, which is identified without knowledge of the surplus function  $\zeta$ , as long as the latter satisfies single crossing (assumption S1), identification of the preference function  $\bar{U}(x, z)$  does require a priori knowledge of the function  $\zeta$ .

### III. Single-Market Identification with Multiple Attributes

This section develops the multivariate analogue of identification results in section II. The strategy follows the same lines. First, a shape restriction on the utility function, analogous to assumption S1, and a consequence of maximization behavior, called “cyclical monotonicity,” will identify the inverse demand: we show that a single type  $\varepsilon(x, z)$  chooses good quality  $z$  at equilibrium. Second, formally, the utility is then recovered from the first-order condition of the consumer’s program. The latter step, however, involves significant difficulties due to the possible lack of differentiability of the endogenous price function.

#### A. Identification of Inverse Demand

##### 1. Shape Restriction

In the one-dimensional case, identification of inverse demand was shown under the single-crossing assumption S1. We noted that the sign of the single-crossing condition is not important for the identification result. Instead, what is crucial is the following, weaker, condition, which is commonly known as the twist condition in the optimal transport literature. The crucial condition, maintained throughout, is assumption S2. As shown in Chiappori, McCann, and Nesheim (2010), it holds when for each distinct pair of consumer types  $(\varepsilon_1, \varepsilon_2)$ , the function  $z \mapsto \zeta(x, \varepsilon_1, z) - \zeta(x, \varepsilon_2, z)$  has no critical point. It is satisfied, for example, in the case  $\zeta(x, \varepsilon, z) = z\varepsilon$ ; in the case  $\zeta(x, \varepsilon, z) = F(x, z\varepsilon)$ , when  $z\varepsilon > 0$  and  $F$  is increasing and convex in its second argument; or in the case  $\zeta(x, \varepsilon, z) = \sum_{k=1}^d F_k(x, \varepsilon_k, z_k)$ , where, for each  $k$  and  $x$ ,  $F_k$  is supermodular in  $(\varepsilon_k, z_k)$ . All these examples are discussed in sections III.C.1 and III.C.2 below.

ASSUMPTION S2 (Twist condition). For all  $x$  and  $z$ , the following hold.

- A. The gradient  $\nabla_z \zeta(x, \varepsilon, z)$  of  $\zeta(x, \varepsilon, z)$  in  $z$  is injective as a function of  $\varepsilon \in \text{Supp}(P_{\varepsilon|x})$ .
- B. The gradient  $\nabla_\varepsilon \zeta(x, \varepsilon, z)$  of  $\zeta(x, \varepsilon, z)$  in  $\varepsilon$  is injective as a function of  $z \in Z$ .

From Gale and Nikaido (1965), it is sufficient that  $D_{xx}^2 \zeta(x, \varepsilon, z)$  be positive definite everywhere for assumption S2 to be satisfied. Alternative sets of sufficient conditions are given in theorem 2 of Mas-Colell (1979).<sup>8</sup> Assumption S2, unlike single crossing, is well defined in the multivariate case, and below we show, using recent developments in optimal transport theory, that it continues to deliver the desired identification in the multivariate case.

<sup>8</sup> Relatedly, Berry, Gandhi, and Haile (2013) propose injectivity results under a gross-substitutes condition.

2. Cyclical Monotonicity

An important implication of assumption EC is that traded quality  $z$  maximizes the joint surplus  $U(\tilde{x}, z) - C(\tilde{y}, z)$ .<sup>9</sup> Let, therefore,  $S(x, \varepsilon, \tilde{y}) := \sup_{z \in Z} [U((x, \varepsilon), z) - C(\tilde{y}, z)]$  be the surplus of a consumer-producer pair  $((x, \varepsilon), \tilde{y})$  at equilibrium. Suppose that consumer  $(x, \varepsilon_0)$   $((x, \varepsilon_1))$  is paired at equilibrium with producer  $\tilde{y}_0$   $(\tilde{y}_1)$  to exchange good quality  $z$ . Then, as shown in the proof of lemma 1 in the appendix, their total surplus is at least as large in the current consumer-producer matching as it would be were they to switch partners. This is a property of the optimal allocation called *cyclical monotonicity*. The total surplus cannot be improved by a cycle of reallocations of consumers and producers. Applied here to cycles of length two, cyclical monotonicity yields

$$\begin{aligned} S(x, \varepsilon_0, \tilde{y}_0) + S(x, \varepsilon_1, \tilde{y}_1) &\geq S(x, \varepsilon_0, \tilde{y}_1) + S(x, \varepsilon_1, \tilde{y}_0) \\ &\geq U((x, \varepsilon_0), z) - C(\tilde{y}_1, z) + U((x, \varepsilon_1), z) - C(\tilde{y}_0, z) \\ &= S(x, \varepsilon_0, \tilde{y}_0) + S(x, \varepsilon_1, \tilde{y}_1), \end{aligned}$$

where the first inequality holds because of cyclical monotonicity and the second inequality holds by definition of the surplus function  $S$  as a supremum. Hence, equality holds throughout in the previous display. Therefore, choice  $z$  maximizes both  $z \mapsto U((x, \varepsilon_0), z) - C(\tilde{y}_0, z)$  and  $z \mapsto U((x, \varepsilon_1), z) - C(\tilde{y}_0, z)$ . It follows that  $\nabla_z \zeta(x, \varepsilon_1, z) = \nabla_z \zeta(x, \varepsilon_0, z)$ . The twist assumption S2 then yields equality of  $\varepsilon_0$  and  $\varepsilon_1$  and the following lemma (proved formally in app. B).

LEMMA 1 (Identification of inverse demand). Under assumptions EC, H, S2(A), and R, if consumers with characteristics  $(x, \varepsilon_0)$  and  $(x, \varepsilon_1)$  consume the same good quality  $z$  at equilibrium, then  $\varepsilon_0 = \varepsilon_1$ .

We see from lemma 1 that identification of inverse demand holds under conditions that are analogous to the scalar case, where the twist condition replaces Spence-Mirrlees as a shape restriction. We also see that the identification proof also relies on a notion of monotonicity. We push this analogy further in section IV and show that the inverse-demand function  $z \mapsto \varepsilon(x, z)$  itself satisfies a generalized form of monotonicity we call “ $\zeta$ -monotonicity,” since its definition involves the function  $\zeta$ .

B. Identification of Marginal Utility

Heuristically, once identification of inverse demand is established and  $\varepsilon(x, z)$  is uniquely defined in equilibrium, the first-order condition of the consumer’s problem  $\sup_{z \in Z} \{\zeta(x, \varepsilon, z) - V(x, z)\}$  delivers identification of

<sup>9</sup> This observation is the basis for the characterization of hedonic models as transferable-utility matching models in Chiappori, McCann, and Nesheim (2010).

marginal utility  $\nabla_z \bar{U}(x, z)$ . However, using the first-order condition presupposes smoothness of the potential  $V$ , hence of the endogenous price function  $z \mapsto p(z)$ . Conditions for differentiability of the potential  $V$  in optimal transportation problems are given by Ma, Trudinger, and Wang (2005). They are applied to identification in hedonic models in Nesheim (2015). However, the conditions in Ma, Trudinger, and Wang (2005) are not transparent, and they are known to be very strong, excluding simple forms of the surplus such as  $S(\tilde{x}, \tilde{y}) := |\tilde{x} - \tilde{y}|^p$  for all  $p \neq 2$ . The remainder of this section, therefore, is devoted to proving a weaker form of differentiability of the price function, which can be used to identify marginal utility from the first-order condition of the consumer's problem.

The consumer's problem yields the expression for indirect utility

$$\sup_{z \in Z} \{\zeta(x, \varepsilon, z) - V(x, z)\} := V^\zeta(x, \varepsilon). \quad (2)$$

Equation (2) defines a generalized notion of convex conjugation, in which the consumer's indirect utility is the conjugate of  $V(x, z) = p(z) - \bar{U}(x, z)$ . This notion of conjugation can be inverted, similarly to convex conjugation, into

$$V^{\zeta\zeta}(x, z) = \sup_{\varepsilon \in \mathbb{R}^d} \{\zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon)\}, \quad (3)$$

where  $V^{\zeta\zeta}$  is called the *double conjugate* of  $V$ . In the special case  $\zeta(x, \varepsilon, z) = z\varepsilon$ , the  $\zeta$ -conjugate simplifies to the ordinary convex conjugate of convex analysis, and convexity of a lower semicontinuous function is equivalent to equality with its double conjugate. Hence, by extension, equality with its double  $\zeta$ -conjugate defines a generalized notion of convexity (see definition 2.3.3 of Villani 2003, 86).

**DEFINITION 1** ( $\zeta$ -convexity). A function  $V$  is called  $\zeta$ -convex if  $V = V^{\zeta\zeta}$ .

We establish  $\zeta$ -convexity of the potential as a step toward a notion of differentiability, in analogy with convex functions, which are locally Lipschitz, and hence almost surely differentiable, by Rademacher's theorem (see, for instance, Villani 2009, theorem 10.8(ii)). It also delivers a notion of  $\zeta$ -monotonicity for inverse demand, discussed in section IV.

**LEMMA 2.** Under assumptions EC and H, the function  $z \mapsto V(x, z)$  is  $P_{z|x}$  almost surely  $\zeta$ -convex, for all  $x$ .

This result provides information only for pairs  $(x, z)$ , where type  $x$  consumers choose good quality  $z$  in equilibrium. In order to obtain a global smoothness result on the potential  $V$ , we need conditions under which the endogenous distribution of good qualities traded in equilibrium is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}^d$ . They include absolute continuity of the distribution of unobserved heterogeneity, additional smoothness conditions on preferences and technology, and the

twist assumption S2(B), which requires the dimension of unobserved heterogeneity to be the same as the dimension of the good quality space, that is,  $d_y = d_z$  (this is relaxed in sec. III.D.1).

ASSUMPTION H'. Assumption H holds, and the distribution of unobserved tastes  $P_{\epsilon|x}$  is absolutely continuous on  $\mathbb{R}^{d_\epsilon}$  with respect to Lebesgue measure for all  $x$ .

ASSUMPTION R' (Conditions for absolute continuity of  $P_{z|x}$ ). The following hold.

1. The Hessian of total surplus  $D_{zz}^2(U(x, \epsilon, z) - C(\tilde{y}, z))$  is bounded above; that is,  $\|D_{zz}^2(U(x, \epsilon, z) - C(\tilde{y}, z))\| \leq M_1$  for all  $x, \epsilon, z$ , and  $\tilde{y}$ , for some fixed  $M_1$ .
2. If the support of  $P_{\epsilon|x}$  is unbounded, for each  $x \in X$ ,  $\|\nabla_z \zeta(x, \epsilon, z)\| \rightarrow \infty$  as  $\|\epsilon\| \rightarrow \infty$ , uniformly in  $z \in Z$ .
3. The matrix  $D_{zz}^2 \zeta(x, \epsilon, z)$  has full rank for all  $x, \epsilon$ , and  $z$ . Its inverse  $[D_{zz}^2 \zeta(x, \epsilon, z)]^{-1}$  has uniform upper bound  $M_0$ ,  $\|[D_{zz}^2 \zeta(x, \epsilon, z)]^{-1}\| \leq M_0$ , for all  $x, \epsilon$ , and  $z$ , for some fixed  $M_0$ .

We then obtain our principal intermediate lemma, which is of independent interest for the theory of hedonic equilibrium. The proof can be found in the online appendix.

LEMMA 3. Under assumptions EC, H', S2, and R', the endogenous distribution  $P_{z|x}$  of qualities traded at equilibrium is absolutely continuous with respect to Lebesgue measure.

Lemma 4, in the online appendix, shows everywhere differentiability of the double conjugate potential  $V^{\zeta\zeta}$ . This does not imply differentiability of  $V$  everywhere, since  $V$  is  $\zeta$ -convex (i.e.,  $V = V^{\zeta\zeta}$ ) only  $P_{z|x}$  almost everywhere. However, combined with lemma 3, this yields approximate differentiability of  $z \mapsto V(x, z)$  as defined in definition 6 in the online appendix. Using uniqueness of the approximate gradient of an approximately differentiable function then yields identification of marginal utility  $\nabla_z \bar{U}(x, z)$  from the first-order condition

$$\nabla_z \zeta(x, \epsilon(x, z), z) = \nabla_{\text{ap},z} V(x, z) = \nabla_{\text{ap}} p(z) - \nabla_z \bar{U}(x, z), \tag{4}$$

where  $\nabla_{\text{ap},(z)}$  denotes the approximate gradient (with respect to  $z$ ) of definition 6, in the online appendix, and where the inverse-demand function  $\epsilon(x, z)$  is uniquely determined, by lemma 1. This yields our main identification theorem.

THEOREM 1. Under assumptions EC, H', S2, and R', the following hold.

1.  $\bar{U}(x, z)$  is nonparametrically identified, in the sense that  $z \mapsto \nabla_z \bar{U}(x, z)$  is the only marginal utility function compatible with the pair  $(P_{xz}, p)$ ;

that is, any other marginal utility function coincides with it,  $P_{z|x}$  almost surely.

2. For all  $x \in X$ ,  $\bar{U}(x, z) = p(z) - V(x, z)$ , and  $z \mapsto V(x, z)$  is  $P_{z|x}$  almost everywhere equal to the  $\zeta$ -convex solution to the problem  $\min_V (\mathbb{E}_z[V(x, z)|x] + \mathbb{E}_\varepsilon[V^\zeta(x, \varepsilon)|x])$ , with  $V^\zeta$  defined in equation (2).

Theorem 1(1) provides identification of marginal utility without any restriction on the distributions of observable characteristics of producers and consumers. The latter may include discrete characteristics. Regularity conditions in assumption R' are satisfied in the cases  $\zeta(x, \varepsilon, z) = \varepsilon'z$  and  $\zeta(x, \varepsilon, z) = \exp(\varepsilon'z)$ , as we discuss in sections III.C.1 and III.C.2, respectively. They preclude bunching of consumers at equilibrium, as shown in lemma 3. The result is driven by the shape restriction (assumption S2) and the strong normalization assumption on the distribution of unobserved heterogeneity. This assumption is inevitable in a single-market identification based on a generalized quantile identification strategy. Section III.D.2 discusses (partial) identification without knowledge a priori of the distribution of unobserved heterogeneity. Theorem 1(2) provides a framework for estimation and inference on the identified marginal utility based on new developments in computational optimal transport (see, for instance, the survey in Peyré and Cuturi 2019).

### C. Special Cases

#### 1. Marginal Utility Linear in Unobserved Taste

A leading special case of the identification result in theorem 1 is the choice  $\zeta(x, \varepsilon, z) = \varepsilon'z$ , where marginal utility is linear in unobservable taste. A natural interpretation of this specification is that each quality dimension  $z_j$  is associated with a specific unobserved taste intensity  $\varepsilon_j$  for that particular quality dimension. Assumptions S2 and R'(2, 3) are automatically satisfied when  $\zeta(x, \varepsilon, z) = \varepsilon'z$ . In addition,  $\zeta$ -convexity reduces to traditional convexity, so that we have the following corollary of theorem 1.

**COROLLARY 1.** Under assumptions EC, H' with  $\zeta(x, \varepsilon, z) = \varepsilon'z$ , and R'(1), the following hold.

1.  $\bar{U}(x, z)$  is nonparametrically identified, in the sense that  $z \mapsto \nabla_z \bar{U}(x, z)$  is the only marginal utility function compatible with the pair  $(P_{xz}, p)$ , that is, any other marginal utility function coincides with it,  $P_{z|x}$  almost surely.
2. For all  $x \in X$ ,  $\bar{U}(x, z) = p(z) - V(x, z)$ , and  $z \mapsto V(x, z)$  is  $P_{z|x}$  almost everywhere equal to the convex solution to the problem  $\min_V (\mathbb{E}_z[V(x, z)|x] + \mathbb{E}_\varepsilon[V^*(x, \varepsilon)|x])$ , where  $V^*$  is the convex conjugate of  $V$ .



A significant computational advantage of corollary 1(2) over the general case is that the potential solves a convex program. The first-order condition (4) also simplifies to  $\varepsilon(x, z) = \nabla_{\text{ap},z} V(x, z)$ , where  $\nabla_{\text{ap},z}$  denotes the approximate gradient with respect to  $z$  (definition 6, in the online appendix). Hence, inverse demand in this case is the approximate gradient of a  $P_{z|x}$  almost surely convex function. This can be interpreted as a multivariate version of monotonicity of inverse demand and assortative matching, since in the univariate case, we recover monotonicity of inverse demand as in section II.

2. Nonlinear Transforms and Random Barten Scales

In the special case of section III.C.1, the curvature of marginal willingness to pay for qualities varies only with observable characteristics but is independent of unobservable type  $\varepsilon$ . Two ways specification  $\zeta(z, \varepsilon, z) = z'\varepsilon$  can be generalized to allow the curvature of marginal utility to vary with unobserved type are the following.

1. Specification  $\zeta(x, \varepsilon, z) = F(x, z'\varepsilon)$  satisfies assumptions S2 and R' when the conditional distribution  $P_{\varepsilon|x}$  of types has bounded support,  $z'\varepsilon > 0$ , and  $F$  is increasing and convex in its second argument (with nonzero derivative). A special case is  $\zeta(x, \varepsilon, z) := \exp(z'\varepsilon)$ . Under this specification, a type  $\varepsilon$  consumer's marginal willingness to pay for quality  $z$  is  $\nabla_z \bar{U}(x, z) + \exp(z'\varepsilon)\varepsilon$ . The curvature of marginal willingness to pay increases with type, so that there are increasing returns to quality.
2. Specification  $\zeta(x, \varepsilon, z) = \sum_{k=1}^d F_k(x, \varepsilon_k, z_k)$  satisfies assumption S2 when  $F_k$  is supermodular in  $(\varepsilon_k, z_k)$  for each  $k$  and  $x$ , and it satisfies assumption R' when  $\partial^2 F_k / \partial z_k^2$  is bounded above and  $\partial^2 F_k / \partial \varepsilon_k \partial z_k$  is bounded below for each  $k$ , uniformly over  $x$ . A special case is  $\zeta(x, \varepsilon, z) = \sum_{k=1}^d F_k(x, z_k \varepsilon_k)$ , in the spirit of random Barten scales, as in Lewbel and Pendakur (2017).

D. Extensions

1. Lower-Dimensional Unobserved Heterogeneity

Our main identification result, theorem 1, is obtained under conditions that force the dimension of unobserved heterogeneity to be the same as the dimension  $d_z$  of the good quality space. The conditions that impose  $d_\varepsilon = d_z$  are assumption H', which requires the distribution  $P_{\varepsilon|x}$  to be absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}^{d_\varepsilon}$ , and assumption S2(B), which requires injectivity of  $z \mapsto \nabla_\varepsilon \zeta(x, \varepsilon, z)$ . In the special case  $\zeta(x, \varepsilon, z) = \varepsilon'z$ , the interpretation of each dimension of  $\varepsilon$  as a quality

dimension-specific taste is appealing. However, the choice of dimension of unobserved heterogeneity remains an arbitrary modeling choice.

In this section, we relax these assumptions and analyze identification with unobserved heterogeneity of lower dimension, including  $d_c = 0$  and  $d_c = 1$ . First, recall that inverse demand is identified in lemma 1 under assumptions EC, H, S2(A), and R, which require only  $d_c \leq d_z$ . We also know from lemma 2 that the potential  $z \mapsto V(x, z)$  is  $P_{z|x}$  almost everywhere  $\zeta$ -convex. It is therefore  $\zeta$ -convex on any open subset of the support of  $P_{z|x}$ , which we show implies local identification. To obtain a global identification result, we need this  $\zeta$ -convexity everywhere.

ASSUMPTION S3 ( $\zeta$ -convexity). The potential  $V$  is  $\zeta$ -convex as a function of  $z$  for all  $x$ .

Unfortunately, this global constraint implies a constraint on the endogenous price function, for which we do not have sufficient conditions in the general case. Under assumption S3, we show differentiability of the potential function  $z \mapsto V(x, z)$ , and hence identification of marginal utility.

THEOREM 2 (Lower-dimensional unobservable heterogeneity).

1. Local identification. Under assumptions EC, H, R, R'(2), and S2(A),
  - a.  $\nabla_z \bar{U}(x, z)$  is identified on any open subset of the support of  $P_{z|x}$ , and
  - b.  $v' \nabla_z \bar{U}(x, z)$  is identified for any vector  $v$  tangent to the support of  $P_{z|x}$ .
2. Global identification. If, in addition, assumption S3 holds,  $\nabla_z \bar{U}(x, z)$  is identified.

Local identification therefore holds under very weak assumptions, as seen in theorem 2(1a). However, in cases with lower-dimensional unobserved heterogeneity, consumer choices may be concentrated on a lower-dimensional manifold, so that there are no open subsets in the support of  $P_{z|x}$ . In such cases, theorem 2(1b) tells us that we can identify marginal willingness to pay only along the support of good attributes actually traded at equilibrium. To illustrate the idea, suppose that consumers are acquiring housing. The latter is differentiated along two dimensions, size and air quality, say. Suppose that consumers with identical observable characteristics  $x$  are heterogeneous along a single scalar dimension of unobserved heterogeneity  $\varepsilon$ . We would then expect equilibrium housing choices to be concentrated on a curve in the (size  $\times$  air quality) space, implicitly defining a scalar quality index that is monotonic in unobserved type  $\varepsilon$ . Our result says that we can identify counterfactual marginal willingness to pay for size and air quality along that curve of observed equilibrium choices only. To obtain global identification with lower-dimensional unobserved

heterogeneity, we need a global  $\zeta$ -convexity assumption on the potential (which implies a shape restriction on the endogenous price function).

We investigate special cases:

1. The case  $\zeta(x, \varepsilon, z) = 0$ . From the first-order condition of the consumer's program, we then have  $\nabla p(z) = \nabla_z U(x, z)$ , so that  $U(x, z)$  is additively separable in  $x$  and  $z$  and the data on matching between consumers and producers cannot inform utility.
2. Scalar unobserved heterogeneity. Suppose that  $d_\varepsilon = 1$  and in addition  $\zeta(x, \varepsilon, z) = \zeta(\varepsilon, z)$ . Denote the (scalar) inverse demand  $z \mapsto \varepsilon(x, z)$ , and assume regularity of all the terms involved. Differentiating the first-order condition of the consumer problem with respect to consumer observable characteristic  $x$  yields

$$D_{xz}^2 \bar{U}(x, z) = -D_{x\varepsilon}^2 \zeta(\varepsilon(x, z), z) \nabla_x \varepsilon(x, z),$$

which is at most of rank 1, since  $D_{x\varepsilon}^2 \zeta(\varepsilon(x, z), z)$  is a  $d_z \times d_\varepsilon$  matrix.

## 2. Partial Identification with Multiple Markets

All identification results so far require fixing the distribution of unobserved consumer heterogeneity a priori. In this section, we derive identifying information from multiple markets, and the possibility of jointly (partially) identifying the utility function  $\bar{U}(x, z)$  and the distribution  $P_{\varepsilon|x}$ . Suppose that  $m_1$  and  $m_2$  index two separate markets, in the sense that producers, consumers, or goods cannot move between markets. Markets differ in the distributions of producer and consumer characteristics  $(P_x^{m_1}, P_y^{m_1})$  and  $(P_x^{m_2}, P_y^{m_2})$ . Suppose, however, that the distribution of unobserved tastes  $P_{\varepsilon|x}$  and the utility function  $U(x, \varepsilon, z) = \bar{U}(x, z) + z'\varepsilon$  are identical in both markets. Both markets are at equilibrium. The equilibrium price schedule in market  $m$  is  $p^m(z)$ . The equilibrium distribution of traded qualities in market  $m$  is  $P_{z|x}^m$ .

Under the assumptions of corollary 1, in each market, we recover a nonparametrically identified utility function  $\bar{U}^m(x, z; P_{\varepsilon|x})$ , where the dependence in the unknown distribution of tastes  $P_{\varepsilon|x}$  is emphasized. For each fixed  $P_{\varepsilon|x}$ , corollary 1 tells us that  $\nabla_z \bar{U}^m(x, z; P_{\varepsilon|x})$  is uniquely determined. In each market, the first-order condition of the consumer's problem is  $\varepsilon(x, z; m) = \nabla_{\text{ap},z} p^m(z) - \nabla_z \bar{U}(x, z; P_{\varepsilon|x})$ . Differencing across markets therefore yields

$$\varepsilon(z, x; m_1) - \varepsilon(z, x; m_2) = \nabla_{\text{ap},z} (p^{m_1}(z) - p^{m_2}(z)), \tag{5}$$

which is an identifying equation for  $P_{\varepsilon|x}$ . The right-hand side of equation (5) is identified, since the price functions are observed. Moreover, for each market  $m$ , the inverse demand  $\varepsilon(x, z; m)$  uniquely determines  $P_{\varepsilon|x}$ , since

it pushes the identified  $P_{z|x}^m$  forward to  $P_{\varepsilon|x}$ . Point identification would require conditions under which the difference  $\varepsilon(z, x; m_1) - \varepsilon(x, z; m_2)$  uniquely determines  $P_{\varepsilon|x}$ , which is beyond the scope of this work.

#### IV. Identification of Nonseparable Simultaneous Equations

The analysis of hedonic equilibrium models in section III motivates a new approach to the identification of nonseparable-simultaneous-equations models of the type  $H(x, z) = \varepsilon$ , where  $x \in \mathbb{R}^d$  is an observed vector of covariates,  $z \in \mathbb{R}^d$  is the vector of dependent variables,  $P_{z|x}$  is identified from the data,  $H$  is an unknown function, and  $\varepsilon \in \mathbb{R}^d$  is a vector of unobservable shocks with distribution  $P_{\varepsilon|x}$ . In the case  $d_\varepsilon = d_z = 1$ ,  $H$  is identified by Matzkin (2003) subject to the normalization of  $P_{\varepsilon|x}$  and monotonicity of  $z \mapsto H(x, z)$  for all  $x$ . This section develops a class of shape restrictions that allows identification of  $H$  in the multivariate case  $1 \leq d_\varepsilon \leq d_z$ .

As in the scalar case, we fix the conditional distribution  $P_{\varepsilon|x}$  of errors a priori. This is justified by the fact that for any vector of dependent variables  $Z \sim P_{z|x}$  and any pair of absolutely continuous error distributions  $(P_{\varepsilon|x}, \tilde{P}_{\varepsilon|x})$ , there is an invertible mapping  $T$  such that  $H(x, Z) \sim P_{\varepsilon|x}$  and  $\tilde{H}(x, Z) := H(x, T(Z)) \sim \tilde{P}_{\varepsilon|x}$  (by McCann 1995), so that  $(H, P_{\varepsilon|x})$  and  $(\tilde{H}, \tilde{P}_{\varepsilon|x})$  are observationally equivalent.

For identification, we rely on a shape restriction that emulates monotonicity in  $z$  of  $H(x, z)$  in the scalar case. This generalized monotonicity notion is inherited from utility-maximizing choices of good quality  $z$  by consumers with characteristics  $(x, \varepsilon)$ . As such, it is indexed by the utility function.

**DEFINITION 2** ( $\zeta$ -monotonicity). Let  $\zeta$  be a function on  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  that is continuously differentiable in its second and third variables and satisfies assumption S2(A). A function  $H$  on  $\mathbb{R}^d \times \mathbb{R}^d$  is  $\zeta$ -monotone if for all  $x$ , there exists a  $\zeta$ -convex function  $z \mapsto V(x, z)$  (see definition 1) such that for all  $z$ ,  $H(x, z) \in \partial_z^\zeta V(x, z)$ , where  $\partial_z^\zeta$  denotes the  $\zeta$ -subdifferential with respect to  $z$  from definition 4, in the online appendix.

Two special cases help clarify the concept of  $\zeta$ -monotonicity:

1. When  $d_\varepsilon = d_z = 1$ , injectivity of  $\varepsilon \mapsto \zeta_z(x, \varepsilon, z)$  implies that  $V^\zeta$  is convex or concave and that  $z \mapsto H(x, z)$  is monotone.
2. When  $d_\varepsilon = d_z$  and  $\zeta(x, \varepsilon, z) = z'\varepsilon$ ,  $V^\zeta = V^*$ , which is convex and therefore locally Lipschitz, hence almost surely differentiable by Rademacher's theorem (Villani 2009, theorem 10.8(ii)), so that  $H(x, z) = \nabla_z V(x, z)$  is the gradient of a convex function.

The class of  $\zeta$ -monotone functions has structural underpinnings as demand functions resulting from the maximization of a utility function

over good qualities  $z \in \mathbb{R}^d$ . Suppose that a consumer with characteristics  $(x, \varepsilon)$  chooses  $z$  on the basis of the maximization of  $\bar{U}(x, z) + \zeta(x, \varepsilon, z) - p(z)$ . Suppose that  $\zeta$  satisfies assumption S2 and that  $V(x, z) := p(z) - \bar{U}(x, z)$  is  $\zeta$ -convex. Then, the demand function  $H$  that satisfies  $\nabla V(x, z) = \nabla_z \zeta(x, H(x, z), z)$ ,  $P_{z|x}$  almost surely, is  $\zeta$ -monotonic by theorem 10.28(b) of Villani (2009, 243). Theorem 3 then shows identification of demand when utility is of the form  $\bar{U}(x, z) + \zeta(x, \varepsilon, z)$  and  $\zeta$  is fixed.

**THEOREM 3** (Nonseparable simultaneous equations). In the simultaneous-equations model  $H(x, z) = \varepsilon$ , with  $z \in \mathbb{R}^d$  and  $x \in \mathbb{R}^d$ , and  $\varepsilon$  following known distribution  $P_{\varepsilon|x}$  the function  $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is identified within the class of measurable  $\zeta$ -monotone functions, for any function  $\zeta$  on  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  that is bounded above, is continuously differentiable in its second and third variables, and satisfies assumptions S2(A) and R'(2), and  $\int \zeta(x, \varepsilon, z) dP_{\varepsilon|x}(\varepsilon) \geq C \in \mathbb{R}$  for all  $x$  and  $z$ .

Theorem 3 is a relatively straightforward application of classical results in optimal transport theory, in particular theorem 10.28 of Villani (2009, 243). Brenier's (1991) polar factorization theorem was, to the best of our knowledge, first used to define multivariate quantile functions by Ekeland, Galichon, and Henry (2012) and Galichon and Henry (2012), with decision-theoretic applications. Carlier, Chernozhukov, and Galichon (2016) and Chernozhukov et al. (2017), both coetaneous with the present paper, apply McCann (1995) to multivariate quantile regression and to multivariate depth, quantiles, ranks, and signs, respectively. This section relies on an extension of these optimal transport results to more general transport costs and interprets it as an identification result, thus extending scalar quantile identification strategies.

If we revisit the two special cases of  $\zeta$ -monotonicity above, we obtain the classical quantile identification of Matzkin (2003) and a result on the identification of nonseparable-simultaneous-equations systems within the class of gradients of convex functions.

1. When  $d_z = d_\varepsilon = 1$ , theorem 3 yields the identification of  $H$  in the system  $\varepsilon = H(x, z)$  when  $z \mapsto H(x, z)$  is monotone.
2. When  $d_\varepsilon = d_z$  and  $\zeta(x, \varepsilon, z) = z'\varepsilon$ , theorem 3 yields the identification of  $H$  in the system  $\varepsilon = H(x, z)$  when  $z \mapsto H(x, z)$  is the gradient of a convex function.

**COROLLARY 2.** In the simultaneous-equations model  $z = G(x, \varepsilon)$ , with  $z \in \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \sim P_{\varepsilon|x}$ , and  $P_{\varepsilon|x}$  a given absolutely continuous distribution on  $\mathbb{R}^d$ , the function  $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is identified within the class of gradients of convex functions of  $\varepsilon$ , for each  $x$ .

Although the previous result is presented as a corollary of theorem 3, it holds under weaker conditions than would be implied by theorem 3 in case  $\zeta(x, \varepsilon, z) = z'\varepsilon$  and is a direct application of the "Main Theorem" in

McCann (1995). The only constraint is the absolute continuity of the distribution of  $\varepsilon$ , so that the outcome vector  $z$  and the covariate vector  $x$  are unrestricted.

Beyond the special cases of corollary 2, we revisit the examples of section III.C.2. First, consider the case of exponential transform  $\zeta(x, \varepsilon, z) := \exp(z'\varepsilon)$ . Given marginal utility  $\bar{U}(x, z) + \exp(z'\varepsilon)$  and price  $p(z)$ , theorem 3 tells us that the solution  $\varepsilon = H(x, z)$  to the consumer's problem  $\nabla p(z) - \nabla_z \bar{U}(x, z) = \exp(z'H(x, z))H(x, z)$  is unique. In the case where consumers are maximizing utility of the form  $\bar{U}(x, z) + \sum_{k=1}^d F_k(x, z_k \varepsilon_k)$ , the corresponding system of differential equations with a unique solution is  $p_{z_k}(z) - \bar{U}_{z_k}(x, z) = f_k(x, z_k H_k(x, z))H_k(x, z)$ , each  $k$ , where  $p_{z_k}$  and  $\bar{U}_{z_k}$  are the partial derivatives with respect to the  $k$ th variable,  $f_k$  is the derivative of  $F_k$  with respect to the second argument, and  $H_k$  is the  $k$ th component of  $H$ .

## V. Discussion

This paper proposed a set of conditions under which utilities and costs in a hedonic equilibrium model are identified from the observation of a single-market outcome. The proof strategy extends Ekeland, Heckman, and Nesheim (2004) and Heckman, Matzkin, and Nesheim (2010; hereafter the EHMN approach) to the case of goods characterized by more than one attribute. The proposed shape restriction on the utility function, called the twist condition, extends the single-crossing condition in the EHMN approach. The proof of identification mirrors that of (one of the strategies in) the EHMN approach. First, inverse demand is identified from the twist condition and cyclical monotonicity (a feature of equilibrium). Then, the first-order condition of the consumer's problem allows the recovery of the utility function, once a suitable form of weak differentiability of the endogenous price function is ensured. The identification proof highlights another parallel with the EHMN approach, which is (generalized) monotonicity of inverse demand. In the scalar case, this generalized monotonicity reduces to monotonicity, whereas in the special case, where utility takes the form  $U((x, \varepsilon), z) = \bar{U}(x, z) + z'\varepsilon$ , inverse demand is the gradient of a convex function. We then show that this generalized form of monotonicity is a suitable shape restriction to identify nonseparable-simultaneous-equations models with a strategy that extends the quantile identification of Matzkin (2003). Most of our results involve fixing the distribution of unobserved consumer heterogeneity a priori, as in the original quantile identification method. Although we provide some discussion of the case, where data from multiple distinct markets can provide additional identifying equations to (partially) identifying  $(\bar{U}(x, z), P_{\varepsilon|x})$  jointly, more research is needed to develop point identification conditions for the latter.

**Appendix A**

**Notation**

Throughout the paper, we use the following notational conventions. Let  $f(x, y)$  be a real-valued function on  $\mathbb{R}^d \times \mathbb{R}^d$ . When  $f$  is sufficiently smooth, the gradient of  $f$  with respect to  $x$  is denoted  $\nabla_x f$ , and the matrix of second-order derivatives with respect to  $x$  and  $y$  is denoted  $D_{xy}^2 f$ . When  $f$  is not smooth,  $\partial_x f$  refers to the subdifferential with respect to  $x$ , from definition 2, in the online appendix, and  $\nabla_{\text{ap},x} f$  refers to the approximate gradient with respect to  $x$ , from definition 6, in the online appendix. The set of all Borel probability distributions on a set  $Z$  is denoted  $\Delta(Z)$ . A random vector  $\varepsilon$  with probability distribution  $P$  is denoted  $\varepsilon \sim P$ , and  $X \sim Y$  means that the random vectors  $X$  and  $Y$  have the same distribution. The product of two probability distributions  $\mu$  and  $\nu$  is denoted  $\mu \otimes \nu$ , and for a map  $f : X \rightarrow Y$  and  $\mu \in \Delta(X)$ ,  $\nu := f\#\mu$  is the probability distribution on  $Y$  defined for each Borel subset  $A$  of  $Y$  by  $\nu(A) = \mu(f^{-1}(A))$ . For instance, if  $T$  is a map from  $X$  to  $Y$  and  $\nu$  a probability distribution on  $X$ , then  $\mu := (\text{id}, T)\#\nu$  defines the probability distribution on  $X \times Y$  by  $\mu(A) = \int_X 1_A(x, T(x)) d\nu(x)$  for any measurable subset  $A$  of  $X \times Y$ . Given two probability distributions  $\mu$  and  $\nu$  on  $X$  and  $Y$ , respectively,  $\mathcal{M}(\mu, \nu)$  denotes the subset of  $\Delta(X \times Y)$  containing all probability distributions with marginals  $\mu$  and  $\nu$ . We denote the inner product of two vectors  $x$  and  $y$   $x'y$ . The Euclidean norm is denoted  $\|\cdot\|$ . The notation  $|a|$  refers to the absolute value of the real number  $a$ , whereas  $|A|$  refers to the Lebesgue measure of set  $A$ . The set of all continuous real-valued functions on  $Z$  is denoted  $C^0(Z)$ , and  $B_r(x)$  is the open ball of radius  $r$  centered at  $x$ . For each fixed  $x \in X$ , the function  $\varepsilon \mapsto V^*(x, \varepsilon) := \sup_{z \in \mathbb{R}^d} \{z'\varepsilon - V(x, z)\}$  is called the *convex conjugate* (also known as the Legendre-Fenchel transform) of  $z \mapsto V(x, z)$ . Still for fixed  $x$ , the convex conjugate  $z \mapsto V^{**}(x, z) := \sup_{\varepsilon \in \mathbb{R}^d} \{z'\varepsilon - V^*(x, \varepsilon)\}$  is called the *double conjugate* or *convex envelope* of the potential function  $z \mapsto V(x, z)$ . According to convex duality theory (see, for instance, Rockafellar 1970, 104, theorem 12.2), the double conjugate of  $V$  is  $V$  itself if and only if  $z \mapsto V(x, z)$  is convex and lower-semicontinuous. The notation  $\psi^\zeta$  refers to the  $\zeta$ -convex conjugate of the function  $\psi$  (definition 3, in the online appendix), and  $\partial_x^\zeta$  refers to the  $\zeta$ -subdifferential with respect to  $x$  (definition 4, in the online appendix).

**Appendix B**

**Proof of Results**

*B1. Proof of Lemma 2*

By definition of  $V^\zeta$ , we have

$$V(x, z) \geq \zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon). \tag{B1}$$

As, by definition of  $\zeta$ -conjugation,  $V^{\zeta\zeta}(x, z) = \sup_\varepsilon [\zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon)]$ , we have

$$V(x, z) \geq V^{\zeta\zeta}(x, z), \tag{B2}$$

by taking a supremum over  $\varepsilon$  in (B1).

Let  $\gamma$  be a hedonic equilibrium probability distribution on  $\tilde{X} \times Z \times \tilde{Y}$ . By assumption EC,

$$\begin{aligned} \zeta(x, \varepsilon, z) - V(x, z) &= U(x, \varepsilon, z) - p(z) = \max_{z \in Z} (U(x, \varepsilon, z) - p(z)) \\ &= \max_{z \in Z} (\zeta(x, \varepsilon, z) - V(x, z)) = V^\zeta(x, \varepsilon) \end{aligned}$$

is true  $\gamma$  almost everywhere. Hence, there is equality in (B1)  $\gamma$  almost everywhere. Hence, for  $P_{z|x}$  almost every  $z$ , and  $\varepsilon$  such that  $(z, \varepsilon)$  is in the support of  $\gamma$ , we have  $V(x, z) = \zeta(x, \varepsilon, z) - V^\zeta(x, \varepsilon)$ . But the right-hand side is bounded above by  $V^{\zeta\zeta}(x, z)$  by definition, so we get  $V(x, z) \leq V^{\zeta\zeta}(x, z)$ . Combined with (B2), this tells us that  $V(x, z) = V^{\zeta\zeta}(x, z)$ ,  $P_{z|x}$  almost everywhere. QED

## B2. Proof of Lemma 1

For a fixed observable type  $x$ , assume that the types  $\tilde{x}_0 := (x, \varepsilon_0)$  and  $\tilde{x}_1 := (x, \varepsilon_1)$  both choose the same good,  $\bar{z} \in Z$ , from producers  $\tilde{y}_0$  and  $\tilde{y}_1$ , respectively.

We want to prove that this implies that the unobservable types are also the same; that is, that  $\varepsilon_0 = \varepsilon_1$ . This property is equivalent to having a map from the good qualities  $Z$  to the unobservable types for each fixed observable type.

Note that  $\bar{z}$  must maximize the joint surplus for both  $\varepsilon_0$  and  $\varepsilon_1$ . That is, setting

$$S(x, \varepsilon, \tilde{y}) = \sup_{z \in Z} (\bar{U}(x, z) + \zeta(x, \varepsilon, z) - C(\tilde{y}, z)),$$

we have

$$S(x, \varepsilon_0, \tilde{y}_0) = \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_0, \bar{z}) \quad (\text{B3})$$

and

$$S(x, \varepsilon_1, \tilde{y}_1) = \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_1, \bar{z}). \quad (\text{B4})$$

By assumption EC, we can apply lemma 1 of Chiappori et al. (2010), so that the pair of indirect utilities  $(V, W)$ , where  $V(\tilde{x}) = \sup_{z \in Z} (U(\tilde{x}, z) - p(z))$  and  $W(\tilde{y}) = \sup_{z \in Z} (p(z) - C(\tilde{y}, z))$ , achieve the dual (Kantorovich) of the optimal transportation problem

$$\sup_{\pi \in \mathcal{M}(P, P)} \int S(\tilde{x}, \tilde{y}) d\pi(\tilde{x}, \tilde{y}),$$

with solution  $\pi$ . This implies, from theorem 1.3 of Villani (2003, 19), that for almost all pairs  $(\tilde{x}_0, \tilde{y}_0)$  and  $(\tilde{x}_1, \tilde{y}_1)$ ,

$$\begin{aligned} V(\tilde{x}_0) + W(\tilde{y}_0) &= S(\tilde{x}_0, \tilde{y}_0), \\ V(\tilde{x}_1) + W(\tilde{y}_1) &= S(\tilde{x}_1, \tilde{y}_1), \\ V(\tilde{x}_0) + W(\tilde{y}_1) &\geq S(\tilde{x}_0, \tilde{y}_1), \\ V(\tilde{x}_1) + W(\tilde{y}_0) &\geq S(\tilde{x}_1, \tilde{y}_0). \end{aligned}$$

We therefore deduce the condition (called the 2-monotonicity condition)

$$S(x, \varepsilon_0, \tilde{y}_0) + S(x, \varepsilon_1, \tilde{y}_1) \geq S(x, \varepsilon_1, \tilde{y}_0) + S(x, \varepsilon_0, \tilde{y}_1),$$



recalling that  $\tilde{x}_0 = (x, \varepsilon_0)$  and  $\tilde{x}_1 = (x, \varepsilon_1)$ . Now, by definition of  $S$  as the maximized surplus, we have

$$S(x, \varepsilon_1, \tilde{y}_0) \geq \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_0, \bar{z}) \tag{B5}$$

and

$$S(x, \varepsilon_0, \tilde{y}_1) \geq \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_1, \bar{z}). \tag{B6}$$

Inserting this, as well as equations (B3) and (B4), into the 2-monotonicity inequality yields

$$\begin{aligned} \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_0, \bar{z}) + \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_1, \bar{z}) \\ \geq S(x, \varepsilon_1, \tilde{y}_0) + S(x, \varepsilon_0, \tilde{y}_1) \\ \geq \bar{U}(x, \bar{z}) + \zeta(x, \varepsilon_1, \bar{z}) - C(\tilde{y}_0, \bar{z}) + \bar{U}(x, \bar{z}) \\ + \zeta(x, \varepsilon_0, \bar{z}) - C(\tilde{y}_1, \bar{z}). \end{aligned}$$

But the left- and right-hand sides of the preceding string of inequalities are identical, so we must have equality throughout. In particular, we must have equality in (B5) and (B6). Equality in (B5), for example, means that  $z$  maximizes  $z \mapsto \bar{U}(x, z) + \zeta(x, \varepsilon_1, z) - C(\tilde{y}_0, z)$ , and so, as  $\bar{z}$  is in the interior of  $Z$  by assumption EC, we have

$$\nabla_z \zeta(x, \varepsilon_1, \bar{z}) = \nabla_z C(\tilde{y}_0, \bar{z}) - \nabla_z \bar{U}(x, \bar{z}). \tag{B7}$$

Since  $\bar{z}$  also maximizes  $z \mapsto \bar{U}(x, z) + \zeta(x, \varepsilon_0, z) - C(\tilde{y}_1, z)$ , we also have

$$\nabla_z \zeta(x, \varepsilon_0, \bar{z}) = \nabla_z C(\tilde{y}_1, \bar{z}) - \nabla_z \bar{U}(x, \bar{z}). \tag{B8}$$

Equations (B7) and (B8) then imply  $\nabla_z \zeta(x, \varepsilon_1, \bar{z}) = \nabla_z \zeta(x, \varepsilon_0, \bar{z})$ , and assumption S2(A) implies  $\varepsilon_1 = \varepsilon_0$ . QED

*B3. Proof of Theorem 1*

*Theorem 1(1).*—Since by lemma 4 in the online appendix,  $V(x, z)$  is approximately differentiable  $P_{z|x}$  almost surely, and since  $\bar{U}(x, z)$  is differentiable by assumption,  $p(z) = V(x, z) + \bar{U}(x, z)$  is also approximately differentiable  $P_{z|x}$  almost surely. Since, by lemma 1, the inverse-demand function  $\varepsilon(x, z)$  is uniquely determined, the first-order condition  $\nabla_z \zeta(z, \varepsilon(x, z), z) = \nabla_{ap} p(z) - \nabla_z \bar{U}(x, z)$  identifies  $\nabla_z \bar{U}(x, z)$ ,  $P_{z|x}$  almost everywhere, as required.

*Theorem 1(2).*—In part 1, we have shown uniqueness (up to location) of the pair  $(V, V^\zeta)$  such that  $V(x, z) + V^\zeta(x, \varepsilon) = \zeta(x, \varepsilon, z)$ ,  $\pi$  almost surely. By theorem 1.3 of Villani (2003, 19), this implies that  $(V, V^\zeta)$  is the unique (up to location) pair of  $\zeta$ -conjugates that solves the dual Kantorovich problem, as required. QED

*B4. Proof of Theorems 2(1a) and 2(2)*

Step 1: Differentiability of  $V$  in  $z$

The objective is to prove that the subdifferential (see definition 2 in the online appendix) at each  $z_0$  is a singleton, which is equivalent to differentiability at  $z_0$ .

We show theorem 2(2). The same method of proof applies on any open subset of the support of  $P_{z|x}$  to yield the proof of theorem 2(1a). From assumption S3,  $V(x, z)$  is  $\zeta$ -convex, and hence locally semiconvex (see definition 5 in the online appendix), by proposition C.2 in Gangbo and McCann (1996). We recall the definition of local semiconvexity from that paper. Now, lemma 1 shows that for each fixed  $z$ , the set

$$\{\varepsilon \in \mathbb{R}^d : V(x, z) + V^\zeta(x, \varepsilon) = \zeta(x, \varepsilon, z)\} := \{f(z)\} \tag{B9}$$

is a singleton. We claim that this means that  $V$  is differentiable with respect to  $z$  everywhere. Fix a point  $z_0$ . We will prove that the subdifferential  $\partial_z V(x, z_0)$  contains only one extremal point (for a definition, see Rockafellar 1970, sec. 18). This will yield the desired result. Indeed, all subdifferentials are closed and convex. Hence, so is the subdifferential of  $V$ . By assumption S3,  $V$  is  $\zeta$ -convex, hence continuous, by the combination of propositions C.2 and C.6(i) in Gangbo and McCann (1996). Hence, as the subdifferential of a continuous function, the subdifferential of  $V$  is also bounded. Hence, it is equal to the convex hull of its extreme points (see Rockafellar 1970, theorem 18.5). The subdifferential of  $V$  at  $z_0$  must therefore be a singleton, and  $V$  must be differentiable at  $z_0$  (theorem 25.1 in Rockafellar 1970 can be easily extended to locally semiconvex functions). Let  $q$  be any extremal point in  $\partial_z V(x, z_0)$ . Let  $z_n$  be a sequence satisfying the conclusion in lemma 3 in the online appendix. Now, as  $V$  is differentiable at each point  $z_n$ , we have the envelope condition

$$\nabla_z V(x, z_n) = \nabla_z \zeta(x, \varepsilon_n, z_n), \tag{B10}$$

where  $\varepsilon_n = f(z_n)$  is the unique point giving equality in equation (B9).

As the sequence  $\nabla_z \zeta(x, \varepsilon_n, z_n)$  converges, assumption R'(2) implies that the  $\varepsilon_n$  remain in a bounded set. We can therefore pass to a convergent subsequence  $\varepsilon_n \rightarrow \varepsilon_0$ . By continuity of  $\nabla_z \zeta$ , we can pass to the limit in equation (B10) and, recalling that the left-hand side tends to  $q$ , we obtain  $q = \nabla_z \zeta(x, \varepsilon_0, z_0)$ . Now, by definition of  $\varepsilon_n$ , we have the equality  $V(x, z_n) + V^\zeta(x, \varepsilon_n) = \zeta(x, \varepsilon_n, z_n)$ . By assumption S3,  $V$  and  $V^\zeta$  are  $\zeta$ -convex, hence continuous, by the combination of propositions C.2 and C.6(i) in Gangbo and McCann (1996). Hence, we can pass to the limit to obtain  $V(x, z_0) + V^\zeta(x, \varepsilon_0) = \zeta(x, \varepsilon_0, z_0)$ . But this means that  $\varepsilon_0 = f(z_0)$ , and so  $q = \nabla_z \zeta(x, \varepsilon_0, z_0) = \nabla_z \zeta(x, f(z_0), z_0)$  is uniquely determined by  $z_0$ . This means that the subdifferential can have only one extremal point, completing the proof of differentiability of  $V$ .

Step 2

Since by step 1,  $V(x, z)$  is differentiable  $P_{z|x}$  almost surely, and since  $\bar{U}(x, z)$  is differentiable by assumption,  $p(z) = V(x, z) + \bar{U}(x, z)$  is also differentiable  $P_{z|x}$  almost surely. Since, by lemma 1, the inverse-demand function  $\varepsilon(x, z)$  is uniquely determined, the first-order condition  $\nabla_z \zeta(x, \varepsilon(x, z), z) = \nabla p(z) - \nabla_z \bar{U}(x, z)$  identifies  $\nabla_z \bar{U}(x, z)$ ,  $P_{z|x}$  almost everywhere, as required. QED

B5. Proof of Theorem 2(1b)

The argument in the proof of theorem 2(2) applies to  $V^{\zeta\zeta}(x, z)$ ; this function is therefore differentiable throughout the support of  $P_{z|x}$ . Now, if  $v$  is tangent to the

support of  $P_{z|x}$  at  $z_0$ , there is a curve  $z(t)$  in  $P_{z|x}$  that is differentiable at  $t_0$ , where  $t_0$  is such that  $z_0 = z(t_0)$  and  $z'(t_0) = v$ . Since  $V^{\zeta\zeta}(x, z) = V(x, z)$  on the support, we have  $V^{\zeta\zeta}(x, z(t)) = V(x, z(t))$ . Since the left-hand side is differentiable as a function of  $t$  at  $t_0$ , the right-hand side must be as well, and

$$\nabla_z V^{\zeta\zeta}(x, z(t)) \cdot v = \nabla_z V^{\zeta\zeta}(x, z(t)) \cdot z'(t_0) = \frac{\partial}{\partial t} [V(x, z(t))] |_{t=t_0}.$$

Therefore,  $p(z) = V(x, z) + \bar{U}(x, z)$  is also differentiable along this curve, and

$$\frac{\partial}{\partial t} [p(z(t))] |_{t=t_0} = \frac{\partial}{\partial t} [V(x, z(t))] |_{t=t_0} + \nabla_z \bar{U}(x, z(t_0)) \cdot v.$$

It then follows from the first-order condition that

$$\begin{aligned} \nabla_z \zeta(x, \varepsilon(x, z), z) \cdot v &= \nabla_z V^{\zeta\zeta}(x, z(t)) \cdot v = \frac{\partial}{\partial t} [V(x, z(t))] |_{t=t_0} \\ &= \frac{\partial}{\partial t} [p(z(t))] |_{t=t_0} - \nabla_z \bar{U}(x, z(t_0)) \cdot v, \end{aligned}$$

which identifies the direction derivative  $\nabla_z \bar{U}(x, z(t_0)) \cdot v$ . QED

*B6. Proof of Theorem 3*

Fix  $x \in \mathbb{R}^d$ , and omit from the notation throughout the proof. Fix a twice continuously differentiable function  $\zeta$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that satisfies assumption S2(A). Assume that there exist two  $\zeta$ -monotonic functions  $H$  and  $\tilde{H}$  such that  $H\#P_z = P_z$ . By the definition of  $\zeta$ -monotonicity (definition 2), there exist two  $\zeta$ -convex functions  $\psi$  and  $\tilde{\psi}$  such that  $H \in \partial^\zeta \psi$  and  $\tilde{H} \in \partial^\zeta \tilde{\psi}$ . By definition of the  $\zeta$ -subdifferential (definition 4 in the online appendix), the existence of these functions implies that  $P_z$  almost surely,  $\psi(z) + \psi^\zeta(H(z)) = \zeta(H(z), z)$  and  $\tilde{\psi}(z) + \tilde{\psi}^\zeta(H(z)) = \zeta(\tilde{H}(z), z)$ . Hence, both  $H$  and  $\tilde{H}$  are solutions to the Monge optimal transport problem with cost  $\zeta$ , which is unique by theorem 10.28 of Villani (2009, 243). It remains to show that the assumptions of theorem 10.28 of Villani (2009) are satisfied. Indeed, by assumption R,  $\zeta$  is differentiable everywhere, hence super-differentiable, verifying assumption 10.28(i); 10.28(ii) is assumption S2, and 10.28(iii) is satisfied under assumption R'(2) by step 1 of the proof of theorem 2(2). Finally, integrating  $\int \zeta(x, \varepsilon, z) dP_{\varepsilon|x}(\varepsilon) > C$  over  $P_{z|x}$  with  $\zeta$  bounded above yields  $\int \int \zeta(x, \varepsilon, z) dP_{z|x}(z) dP_{\varepsilon|x}(\varepsilon)$  finite, which is the remaining condition for theorem 10.28 of Villani (2009) to hold. The result follows. QED

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